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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME I.

SESSION 1883.

WILLIAMS AND NORGATE,
14 HENRIETTA STREET, COVENT GARDEN, LONDON; AND
20 SOUTH FREDERICK STREET, EDINBURGH.
1894.

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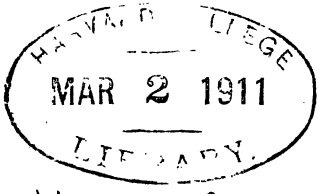

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INDEX.

PAGE

CHRYSTAL, G.
Present fields of mathematical research [Title], - 3

FRASER, A. Y.
The fundamental notions of the Differential Calculus [Title], - - - - - - 129

KNOTT, C. G.
An account of Newton's "Opticks" [Title], - - 129
Plücker's first equation connecting the singularities of curves, - - - - - - 130
Some notes on Quaternions [Title], - - - 136

LIST OF MEMBERS, - - - - - - 137

MACKAY, J. S.
The triangle and its six scribed circles, - - 4
Notation, - - - - - 6
§ 1. Centroid, - - - - - 13
§ 2. Circumcentre, - - - - - 32
§ 3. Incentre, - - - - - 39
§ 4. Excentres, - - - - - 47
§ 5. Orthocentre, - - - - - 60
§ 6. Euler's line, - - - - - 97
§ 7. Relations among radii, - - - 104
§ 8. Area, - - - - - - 117

MUIR, T,
New proof of Professor Tait's problem of arrangement [Title], - - - - - - 129
Some short notes of interest to Teachers [Title], - 129

Digitized by Google

PAGE

MUNN, D.
The nine-point circle [Title], - - - - 129
Some theorems on radical axes [Title], - - - 136

OFFICE-BEARERS, - - - - - - - 3

ERRATA.

PAGE 7. Last line but one, W. J. should be J. W.

„ 18. Sixth line, MM should be MM′.

„ 23. Last line but one, maximum should be minimum.

„ 48. The following historical note in reference to (5) has been omitted :—

As regards the expressions for s, s_1, s_2, s_3 some are given in Heron's theorem regarding the area of a triangle; some are given by Euler in *Novi Commentarii Academiae...Petropolitanae* for the years 1747–8, I. 49–66 (1750). All except those where D, E, F occur are given in the *Ladies' Diary* for 1759; all are given by T. S. Davies in the *Ladies' Diary* for 1835, pp. 51–2.

Six of the expressions for a, b, c are given by T. S. Davies in the *Ladies' Diary* for 1835, p. 52.

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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

FIRST SESSION, 1883.

The EDINBURGH MATHEMATICAL SOCIETY owes its origin to the suggestion of the late Mr A. Y. FRASER, at that time one of the mathematical masters in George Watson's College, Edinburgh. With him were closely associated Mr A. J. G. BARCLAY, now of Glasgow High School, and Dr C. G. KNOTT of the University of Edinburgh. The following is the circular which they issued to those persons throughout the kingdom whom they deemed likely to take an interest in such a Society:

EDINBURGH UNIVERSITY,
January 23, 1883.

DEAR SIR,

We, the undersigned, beg to call your attention to the following proposal, in the hope that you will find it in your power to give it your support:

It is proposed to establish, primarily in connection with the University, a Society for the mutual improvement of its members in the Mathematical Sciences, pure and applied.

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Amongst the methods by which this object might be attained may be mentioned: Reviews of works both British and Foreign, historical notes, discussion of new problems or new solutions, and comparison of the various systems of teaching in different countries, or any other means tending to the promotion of Mathematical Education.

It is suggested that the Society be formed, in the first instance, of all those who shall give in their names on or before February 2, 1883, and who are

(1) present or former students in either of the Advanced Mathematical Classes of Edinburgh University,

(2) Honours Graduates in any of the British Universities, or

(3) recognised Teachers of Mathematics;

and that, after the above mentioned date, members be nominated and elected by ballot in the usual manner.

It may be added that Professors Tait and Chrystal have expressed themselves as highly favourable to the project, as one that may lead to important results.

If there are any of your friends who might take an interest in the Society, kindly inform them of its objects, and invite them to attend the Preliminary Meeting, to be held in the MATHEMATICAL CLASS ROOM here, on Friday, February 2, 1883, at Eight P.M., at which meeting your presence is respectfully requested.

We are,

Yours faithfully,

CARGILL G. KNOTT, D.Sc. (Edin.), F.R.S.E.
A. J. G. BARCLAY, M.A. (Edin.).
A. Y. FRASER, M.A. (Aberdeen).

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Preliminary Meeting, February 2nd, 1883.

C. G. KNOTT, Esq., D.Sc., F.R.S.E., in the Chair.

At this meeting it was agreed to institute a Society to be called "The Edinburgh Mathematical Society."

The following Office-Bearers were elected:—

President—Mr JOHN S. MACKAY, M.A., F.R.S.E.

Vice-President—Mr ALEXANDER MACFARLANE, M.A., D.Sc., F.R.S.E.

Secretary and Treasurer—Mr CARGILL G. KNOTT, D.Sc., F.R.S.E.

Committee.

Messrs R. E. ALLARDICE, M.A.; A. J. G. BARCLAY, M.A.; A. Y. FRASER, M.A.; W. J. MACDONALD, M.A.

The Committee were instructed to draw up a constitution for the Society, and to submit the same at the first ordinary meeting.

First Meeting, March 12th, 1883.

JOHN S. MACKAY, Esq., M.A., F.R.S.E., President, in the Chair.

Professor CHRYSTAL, University of Edinburgh, gave an introductory address on "Present Fields of Mathematical Research."

Thereafter, the Society proceeded to the discussion of the draft constitution submitted by the Committee.

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Second Meeting April 13th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., F.R.S.E., President, in the Chair.

The Triangle and its Six Scribed Circles.

By JOHN STURGEON MACKAY, M.A.

[When the Edinburgh Mathematical Society was founded, it was deemed impracticable, from the expense that would be involved, to print the papers read at its meetings. It was, however, resolved that copies of them should be deposited with the Secretary, and that these copies should as far as possible be made accessible to the members of the Society. Want of leisure during the first session prevented me from doing my part in carrying out this intention of the Society, and committing to writing the paper which was delivered at the second meeting. During its second session, the Society resolved to print its proceedings in whole or in abstract; and a beginning was made with the second volume, the first being left over for future consideration, as the cost of printing absorbed nearly the entire income of the Society.

Some time afterwards, through the exertions of the late Mr A. Y. Fraser, who was then Secretary of the Society, and whose opening career was cut short by untimely death, a sum of money was raised by voluntary subscription to defray the expense of printing Vol. I. Seeing that it was desired by others as well as by myself to have the collection of properties of the triangle as ample as possible, I obtained the consent of the Society to delay the publication of the volume. This delay has been unduly prolonged, and no one but myself is to be blamed in connection with it.

The collection, divided into twenty sections, which has resulted is a tolerably large one, filling somewhat more than 1600 quarto pages of manuscript. Owing to the impossibility of printing all this without a large expenditure of money, that portion has been selected which best corresponds with what was communicated to the Society. The six scribed circles of the title were the inscribed, the escribed,

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the circumscribed, and, as I called it then, the medioscribed (or nine point) circles. Part of the section treating of the nine point circle was printed in abstract in Vol. II. of the *Proceedings*, and the rest, considerably enlarged, in Vol. XI. What is here printed consists of the first section, with the principal properties of some other sections incorporated in it, and a statement of the notation which is employed throughout all the sections. The remainder of the collection will, unless it is anticipated somewhere else, see the light gradually, I hope, in the *Proceedings* of the Society.

This fragmentary memoir would have appeared more original if I had left without note or comment all those properties which I had discovered for myself, and it would certainly have looked more learned if I had furnished references to all the authorities who have discussed various parts of the subject. Unless there was reason to the contrary, as, for example, when a new mode of proof was given, I have noted simply the first discoverer of a theorem, and have suppressed all mention of subsequent discoverers. I have not concerned myself with problems, with questions of loci, or with trigonometrical expressions relating to the triangle, for the simple reason that I wished to keep the memoir within manageable bounds. The same reason has also led me to defer consideration of the properties of the triangle, in which other conic sections than the circle are involved.

The research which has been necessary to render the historical notes scattered throughout the whole collection worthy of attention could not, in the limited leisure at my disposal, have been undertaken except with the aid of many friends. I gladly seize the present opportunity of recording my gratitude for many favours to my colleagues of the Edinburgh Mathematical Society, Messrs Alison, Allardice, Gibson, Harvey, Macdonald, Pressland, and, in particular, the late Mr A. Y. Fraser; to Messrs Anderson, Langley, and Tucker in England; to Professor Neuberg in Belgium; to Dr Grebe and Professor Fuhrmann in Germany: and to Messrs Aubert, Lemoine, De Longchamps, D'Ocagne, and Poulain in France. For two successive summers Messrs Nony and Vuibert put a complete copy of their *Journal de Mathématiques Élémentaires* in my hands, with liberty to make such extracts from it as I pleased; and Mr De Longchamps, with the courtesy of his nation, has several times placed his whole library at my disposal, even when he was absent from Paris.]

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NOTATION

A few preliminary words are necessary in respect to the following scheme of notation.

In the first place the notation should be uniform, that is to say, when a definite point connected with the triangle is under consideration it should always be denoted by the same letter, and not by one letter at one time and by another letter at another. But the converse practice of never using a letter which has denoted one special point to denote any other point need not be carried out, and indeed cannot be, unless the number of properties investigated be very small, or unless recourse be had to letters ornamented with an intolerable number of suffixes or accents. To illustrate by an example. When the circumcentre of a triangle ABC has to be mentioned, it is invariably called O, but in cases where the circumcentre does not come into consideration at all, there is no adequate reason why O should not be used to denote some other point such as the point of concurrency of three angular transversals.

In the second place capital letters, loaded or not as the case may be with an accent or a suffix or even with both, should denote points, and small letters should denote lines. I am aware that such a rule is not adhered to in some of the best geometrical treatises, for instance those of Chasles who constantly uses both capitals and small letters to denote points. The extremely prevalent, though not universal, practice of denoting the radius of the circumcircle by R shows that one deviation at least must be made from the rule if accordance with general usage is to be secured. For the distances between the incentre, the excentres, and the vertices of a triangle Greek letters have been employed. This was found to be unavoidable, if conciseness was to be aimed at. These Greek letters are the notation adopted by the discoverers of not a few of the properties* connected with the incentre, etc., and I could not invent a better.

In the third place the notation should conform as far as may be to that actually in use among geometers. I have had this consideration continually before me, and it has caused me a world of trouble. For the notation has been changed, rechanged, and changed again; and though I dare not hope that the compromise which has been

* The properties referred to are not contained in the section which is here printed.

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come to will commend itself to everybody as the best in the circumstances, yet if it were needful I could support with a considerable weight of authority drawn from one country or another, the designation of every important point to which objection might be taken.

POINTS.

A, B, C, = vertices of the fundamental triangle.
When the sides of triangle ABC are spoken of, they are understood to be taken in the order BC, CA, AB.

A', B', C' = mid points of the sides BC, CA, AB.

A_∞, B_∞, C_∞ = harmonic conjugates of A', B', C', for* BC, CA, AB.

A_1, B_1, C_1
A_2, B_2, C_2
... ...
= vertices of some triangle related to ABC, for example, anticomplementary, inscribed or circumscribed.

D, E, F = points of contact of sides with incircle.
= points of intersection of sides with transversal.
= points of intersection of sides with angular transversals.
= projections of a point on the sides.

D_1, E_1, F_1
D_2, E_2, F_2
D_3, E_3, F_3
= points of contact of sides with { 1st excircle. 2nd excircle. 3rd excircle.

D', E', F' = harmonic conjugates of D, E, F for BC, CA, AB.
= those points in A'B'C' which correspond to D, E, F in ABC.
Similarly for D_1', E_1', F_1', etc.

G, G_a, G_b, G_c = centroids of ABC, HCB, CHA, BAH.

G_0, G_1, G_2, G_3 = centroids of $I_1I_2I_3$, II_3I_2, I_3II_1, I_2I_1I.

---

* Following the example of Mr W. J. Russell, I have used the word "for" instead of the phrase "with respect to."

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Γ	= point of concurrency of AD, BE, CF.
Γ_1, Γ_2, Γ_3	= points of concurrency of AD_1, BE_1, CF_1, and so on.
	These four points are frequently called the Gergonne points of ABC.
H	= orthocentre of ABC.
I	= incentre of ABC.
I_1, I_2, I_3	= 1st, 2nd, 3rd excentres of ABC.
J	= incentre of $A_1B_1C_1$, the triangle anticomplementary to ABC.
J_1, J_2, J_3	= 1st, 2nd, 3rd excentres of $A_1B_1C_1$.
	These four points are frequently called the Nagel points of ABC.
K	= insymmedian point of ABC (Lemoine's point).
K_1, K_2, K_3	= 1st, 2nd, 3rd exsymmedian points of ABC.
L, M, N	= feet of internal angular bisectors of ABC.
	= projections of the symmedian point on the sides of ABC.
L', M', N'	= feet of external angular bisectors of ABC.
L_1, M_1, N_1	= projections of the first exsymmedian point on the sides of ABC.
	Similarly for L_2, M_2, N_2 etc.
O	= circumcentre of ABC.
	= point of intersection of three concurrent lines.
O'	= circumcentre of A'B'C', or nine-point centre of ABC.
O_a, O_b, O_c	= circumcentres of HCB, CHA, BAH.
O_0, O_1, O_2, O_3	= circumcentres of $I_1I_2I_3$, II_3I_2, I_3II_1, I_2I_1I.
O_1, O_2, O_3	= circumcentres of ΩCA, ΩAB, ΩBC.
O_1', O_2', O_3'	= circumcentres of $\Omega'CA$, $\Omega'AB$, $\Omega'BC$.
O, O'	= pairs of isogonal points with respect to ABC.

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P, Q
P, P′
Q, Q′ } = pairs of special points.

R, S, T = points where the perpendiculars of a triangle meet the circumcircle.
= feet of the insymmedians.
= projections of Ω on BC, CA, AB.

R′, S′, T′ = feet of the exsymmedians.
= projections of Ω′ on BC, CA, AB.

T, T_1, T_2, T_3 = points of contact of the nine-point circle with the incircle and the excircles.
= centres of the four Taylor circles.

U, U′ = ends of that diameter of the circumcircle which is perpendicular to BC. U is on the opposite side of BC from A.
Similarly for V, V′ and W, W′.

U, V, W = points in which concurrent lines from A, B, C meet the circumcircle ABC.
= mid points of AH, BH, CH.

X, Y, Z = feet of the perpendiculars from A, B, C.

X′, Y′, Z′ = harmonic conjugates of X, Y, Z for BC, CA, AB.
= mid points of YZ, ZX, XY.

Ω, Ω′ = Brocard points of ABC.

Regarding the notation which may be adopted for many other important points connected with the triangle, such as the isogonic and isodynamic centres, the centres of Brocard's circle and Lemoine's first circle (triplicate ratio), Steiner's point, Tarry's point, and so on, no suggestion is offered for the present. Those who wish to see the notations which have been employed may be referred to Mr De Longchamps' *Journal de Mathématiques Élémentaires* and to *Mathesis*.

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LINES.

$a,\ b,\ c$ = the sides BC, CA, AB of ABC.

$a,\ \beta,\ \gamma$ = AI, BI, CI.

$a_1,\ \beta_1,\ \gamma_1$ = AI_1, BI_1, CI_1.

$a_2,\ \beta_2,\ \gamma_2$ = AI_2, BI_2, CI_2.

$a_3,\ \beta_3,\ \gamma_3$ = AI_3, BI_3, CI_3.

$a_1 - a,\ \beta_2 - \beta,\ \gamma_3 - \gamma$ = $I I_1$, $I I_2$, $I I_3$.

$a_2 + a_3,\ \beta_3 + \beta_1,\ \gamma_1 + \gamma_2$ = I_2I_3, I_3I_1, I_1I_2.

$d,\ e,\ f$ = the sides EF, FD, DE of DEF.
Similarly for the sides of $D_1E_1F_1$, etc.

$h_1,\ h_2,\ h_3$ = the perpendiculars AX, BY, CZ.

$h_1',\ h_2',\ h_3'$ = the segments AH, BH, CH of the perpendiculars.

$h_1'',\ h_2'',\ h_3''$ = the segments HX, HY, HZ of the perpendiculars.

$k_1,\ k_2,\ k_3$ = the perpendiculars OA', OB', OC' from the circumcentre.

$l_1,\ l_2,\ l_3$ = the internal angular bisectors of A, B, C.

$\lambda_1,\ \lambda_2,\ \lambda_3$ = the external angular bisectors of A, B, C.

$m_1,\ m_2,\ m_3$ = the medians from A, B, C.

$n_1,\ n_2,\ n_3$ = the insymmedians from A, B, C.

$\nu_1,\ \nu_2,\ \nu_3$ = the exsymmedians from A, B, C.

$p_1,\ p_2,\ p_3$ = perpendiculars from any point on BC, CA, AB.

r = radius of the incircle.

$r_1,\ r_2,\ r_3$ = radii of the 1st, 2nd, 3rd excircles.

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R	= radius of the circumcircle.
R_1, R_2, R_3	= radii of the circumcircles of ΩCA, ΩAB, ΩBC.
R_1', R_2', R_3'	= radii of the circumcircles of $\Omega'CA$, $\Omega'AB$, $\Omega'BC$.
ρ	= radius of the incircle of XYZ,
ρ_1, ρ_2, ρ_3	= radii of the 1st, 2nd, 3rd excircles of XYZ.
s	= semiperimeter of ABC.
s_1, s_2, s_3	$= s-a,\ s-b,\ s-c.$
x, y, z	= the sides YZ, ZX, XY of XYZ.

AREAS.

Δ, Δ_a, Δ_b, Δ_c	= ABC, HCB, CHA, BAH.
Δ_0, Δ_1, Δ_2, Δ_3	$= I_1I_2I_3,\ II_2I_3,\ I_3II_1,\ I_2I_1I.$

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SECTION I.

§ 1. CENTROID.

§ 2. CIRCUMCENTRE.

§ 3. INCENTRE.

§ 4. EXCENTRES.

§ 5. ORTHOCENTRE.

§ 6. EULER'S LINE.

§ 7. RELATIONS AMONG RADII.

§ 8. AREA.

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§ 1. Centroid.

*The medians of a triangle are concurrent.**

Figure 1.

Let the medians BB', CC' cut each other at G; join AG, and let it cut BC at A'.

Produce AA' to L, making GL equal to GA, and join BL, CL.

Because	C'G bisects AB and AL,
therefore	C'G is parallel to BL.
Similarly	B'G „ „ „ CL;
therefore	BLCG is a parallelogram;
therefore	A' is the mid point of BC.

This theorem may be proved in many other ways.

Def.—The point G is called sometimes the *centre of gravity* † of the triangle ABC; sometimes the *centre of mean distances* ‡ of the points A, B, C; and more frequently now the *centroid* § of the triangle ABC.

The simplest construction for obtaining G by means of the ruler and the compasses is the following ‖ :—

With B as centre and AC as radius describe a circle; with C as centre and AB as radius describe a second circle cutting the former below the base at D. Join DC and produce it to meet the second circle at E.

AD and BE intersect at the centroid G.

(1) $A'G = \frac{1}{2}AG = \frac{1}{3}AA'$.

Hence the centroid of a triangle may be found by drawing any median and trisecting it; and if two (or a series of) triangles have the same vertex and the same median drawn from that vertex, they have the same centroid.

---

* Archimedes, *De planorum æquilibriis*, I. 13, 14.

† Archimedes.

‡ Carnot, *Géométrie de Position*, p. 315 (1803), and Lhuilier, *Élémens d'Analyse*, p. X. (1809).

§ This expression was suggested by T. S. Davies in 1843 in the *Mathematician* I. 58. It had been used by Dr Hey in 1814 to designate another point.

‖ Mr E. Lemoine in the Report (second part) of the 21st session of the *Association Française pour l'avancement des sciences*, p. 77 (1892).

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(2) Triangle $GBC = GCA = GAB = \frac{1}{3}ABC$.

(3) The sides of triangle A'B'C' are respectively parallel to those of ABC; hence these triangles are directly similar.

Also, since the lines AA', BB', CC' joining corresponding vertices are concurrent at G, triangles ABC, A'B'C' are homothetic, and G is their homothetic centre.

DEF.—Triangles such as the fundamental triangle ABC, and that formed by joining the feet of its medians have in recent years received the following names:—

A'B'C' is the *complementary* triangle of ABC.
ABC „ „ *anticomplementary* „ „ A'B'C'.

These names are applied also to corresponding points * in such triangles. Thus if P be any point in or outside of triangle A B C, and P' be the corresponding point in or outside of triangle A'B'C',

P' is the *complementary* point of P,
P „ „ *anticomplementary* „ „ P'.

(4) If $A_1B_1C_1$ be the triangle formed by drawing through A, B, C parallels to the opposite sides of triangle ABC,

A B C is the complementary triangle of $A_1B_1C_1$,
$A_1B_1C_1$ „ „ anticomplementary „ „ A B C.

FIGURE 2.

(5) The fundamental triangle ABC is directly similar to the triangles cut off from it by the sides of its complementary triangle, AC'B', C'BA', B'A'C.

(6) The centroid of the fundamental triangle is the centroid of the complementary triangle; the centroid of the complementary triangle is the centroid of its complementary triangle; and so on.

(7) All straight lines parallel to the base of a triangle and terminated by the other sides are bisected by the median to the base.

---

* See Mr Emile Vigarié's articles *Sur les Points Complémentaires* in *Mathesis*, VII., 5-12, 57-62, 84-9, 105-110 (1887).

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Hence, if EF, GH, KL ... be parallel to BC, the points E, G, K ... being on AC, and F, H, L ... on AB, the intersections of

BE, CF; BG, CH; BK, CL; FG, EH; FK, EL ..... will all lie on the median * from A.

(8) If two triangles have the same base, the straight line which joins their vertices is parallel to and three times as long as the straight line which joins their centroids.

(9) If G be any point in the plane of ABC, and G_a, G_b, G_c be the centroids of triangles GBC, GCA, GAB, triangle $G_aG_bG_c$ is directly similar † to triangle ABC.

(10) If P be any point on the circumcircle of ABC, the centroids of the four triangles PBC, PCA, PAB, ABC are concyclic.‡

For if the centroids of these triangles be denoted by D, E, F, G respectively, the quadrilateral DEFG has its sides DE, EF, FG, GD respectively parallel to BA, CB, PC, AP, and one-third as long.

Mr Griffiths states § that if the circle on which the four centroids lie be called the centroid-circle of the quadrangle ABCP, it may be shown that the centroid-circles of the five quadrangles that can be formed from five concyclic points will also have their centres on the circumference of another circle of one-third the radius of the first.

Townsend gives the following generalisation || of (10):

If A, B, C, D, E, F, etc., be the position of any number (n) of equal masses distributed in space, G that of their centre of gravity, and A′, B′, C′, D′, E′, F′, etc., those of the centres of gravity of their n different groups of $(n-1)$; then always the two systems of n points A, B, C, D, E, F, etc., and A′, B′, C′, D′, E′, F′, etc., are similar, oppositely placed with respect to each other, have G for their centre of similitude, and $(n-1):1$ for their ratio of similitude.

The truth of this is evident, for the several lines AA′, BB′, CC′, DD′, EE′, FF′, etc., all connect through G, and are then divided internally in the common ratio of $(n-1):1$.

---

* Jacobi, *De Triangulorum rectilineorum proprietatibus*, pp. 5-6 (1825).

† Professor R. E. Allardice.

‡ Mr J. Griffiths, in *Mathematical Questions from the Educational Times*, V. 92 (1866).

§ *Notes on the Geometry of the Plane Triangle*, p. 65 (1867).

|| *Mathematical Questions from the Educational Times*, V. 92 (1866).

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DEF.—If the vertex A of a triangle ABC be joined to any point D in the base, the fourth harmonic ray to AB, AD, AC is found by dividing BC externally at D′ in the ratio BD : CD, and joining AD′.

When the point D is the mid point of BC, namely A′, the fourth harmonic ray to AB, AA′, AC is the line through A parallel to BC, and it may be denoted by AA_∞.

Similarly, the line through B parallel to CA will be the fourth harmonic ray to BC, BB′, BA, and may be denoted by BB_∞; the line through C parallel to AB will be the fourth harmonic ray to CA, CC′, CB, and may be denoted by CC_∞.

If therefore AA′, BB′, CC′ be called the *internal medians* of triangle ABC, then AA_∞, BB_∞, CC_∞ may be called the *external medians*.

(11) The six medians, internal and external, of a triangle meet three and three in four points, which are the centroid and the points anticomplementary to the vertices of the triangle, namely G, A_1, B_1, C_1.

FIGURE 2.

DEF.—The points A_1, B_1, C_1, G form a *tetrastigm* (a system of four points, no three of which are collinear), and the three pairs of opposite connectors,

$$A_1G,\ B_1C_1\ ;\ B_1G,\ C_1A_1\ ;\ C_1G,\ A_1B_1$$

meet in A ; B ; C,

which are the *centres* of the tetrastigm, and ABC is the *central triangle* of the tetrastigm.

If ABCG be the tetrastigm, the points A′, B′, C′ are its centres, and A′B′C′ its central triangle.

(12) *If in the internal median AA′ of triangle ABC any point M be taken, and MP, MQ be drawn perpendicular to AC, AB, then MP, MQ are inversely proportional to AC, AB.*

FIGURE 3.

Join MB, MC.

Then $AMB = AMC$;

therefore $AB \cdot MQ = AC \cdot MP$;

therefore $AB : AC = MP : MQ$.

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(13) *If in the external median AA_∞ of triangle ABC any point M′ be taken, and M′P′, M′Q′ be drawn perpendicular to AC, AB, then MP′, M′Q′ are inversely proportional to AC, AB.*

FIGURE 4.

Join M′B, M′C.

Then $\quad AM'B = AM'C;$

therefore $\quad AB \cdot M'Q' = AC \cdot M'P';$

therefore $\quad AB : AC = M'P' : M'Q'.$

(14) If from G the centroid of ABC there be drawn p_1, p_2, p_3 perpendicular to BC, CA, AB, then

$$BC : CA : AB = \frac{1}{p_1} : \frac{1}{p_2} : \frac{1}{p_3}.$$

(15) If from G the centroid of ABC there be drawn p_1', p_2', p_3' perpendicular to B_1C_1, C_1A_1, A_1B_1, then

$$BC : CA : AB = \frac{1}{p_1'} : \frac{1}{p_2'} : \frac{1}{p_3'}.$$

(16) *If the vertex A of the triangle ABC falls on the base BC, the centroid G of the three collinear points A, B, C is found by the construction indicated in* (1):

Bisect BC in A′, and divide AA′ at G so that

$$AG : A'G = 2 : 1.$$

In this case the sum of the distances from the centroid of the points on one side of it is equal to the distance from the centroid of the point on the opposite side.

FIGURE 5.

For
$$\begin{aligned} AG + BG &= (AA' - GA') + (BA' - GA'), \\ &= AA' + BA' - 2GA', \\ &= AA' + CA' - GA, \\ &= CA - GA, \\ &= CG. \end{aligned}$$

(17) If ABC be a triangle, G its centroid, and A′, B′, C′, G′ the projections of A, B, C, G, on any straight line XY, then G′ is the centroid of the three collinear points A′, B′, C′.

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(18) LEMMA.*

If a straight line BC be divided internally at M so that

$$BM : CM = n : m$$

and if from B, M, C perpendiculars BB', MM', CC', be drawn to any straight line XY, then

$$(m+n)MM' = mBB' \pm nCC'$$

the upper sign being taken when B and C are on the same† side of XY, and the lower when they are on opposite sides of XY.

FIGURE 6.

Join BC' meeting MM' in N.

The triangles BB'C, NM'C are similar;

therefore $BB' : NM' = BC' : NC'$
$= BC : MC$
$= m+n : m$;

therefore $mBB' = (m+n)NM'$.

The triangles BCC', BMN are similar;

therefore $CC' : MN = BC : BM$,
$= m+n : n$;

therefore $nCC' = (m+n)MN$.

Hence $mBB' + nCC' = (m+n)NM' + (m+n)MN$
$= (m+n)MM'$.

(19) *The distance of the centroid of a triangle from any straight line is an arithmetic mean between the distances of the vertices from the same straight line.*

FIGURE 7.

Let AM be the median from A, and G the centroid.

Take A', B', C', G', M' the projections of A, B, C, G, M on any straight line XY.

---

* Lhuilier, *Élémens d'Analyse*, pp. 1-2 (1809).

† The figure and demonstration refer only to this case. The other case and the consideration of what happens when BC is divided externally are left to the reader.

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Because $BM : CM = 1 : 1$,
therefore $BB' + CC' = 2MM'$.
Because $AG : MG = 2 : 1$,
therefore $AA' + 2MM' = 3GG'$;
therefore $AA' + BB' + CC' = 3GG'$.

The figure and demonstration refer only to the case when A, B, C are all on the same side of XY. If A and B be on the same side of XY, and C on the opposite side, the result will be

$$AA' + BB' - CC' = 3GG'.$$

When XY passes through the centroid G, the sum of the distances from XY of the vertices on one side of it is equal to the distance from XY of the vertex on the opposite side.

For a very full account of the properties of the centre of mean distances see the preliminary dissertation in Lhuilier's *Élémens d'Analyse* (1809), and Townsend's *Modern Geometry of the Point, Line, and Circle*, I. 117-143 (1863).

(20) *The sum of the squares of the distances of the vertices of a triangle from any point is equal to the sum of the squares of their distances from the centroid increased by three times the square of the distance between the point and the centroid.**

FIGURE 8.

Let G be the centroid of ABC, and P any other point. Join PG, and on it draw perpendiculars from A, B, C.

Then
$$AP^2 = AG^2 + PG^2 - 2PG \cdot A'G,$$
$$BP^2 = BG^2 + PG^2 + 2PG \cdot B'G,$$
$$CP^2 = CG^2 + PG^2 - 2PG \cdot C'G;$$
therefore
$$AP^2 + BP^2 + CP^2 = AG^2 + BG^2 + CG^2 + 3PG^2 - 2PG(A'G - B'G - C'G),$$
$$= AG^2 + BG^2 + CG^2 + 3PG^2,$$
since
$$A'G - B'G - C'G = 0.$$

(21) If a circle be described with G as centre, and any radius, and any two points P, Q be taken on its circumference †

$$AP^2 + BP^2 + CP^2 = AQ^2 + BQ^2 + CQ^2.$$

---

* This is a particular case of a more general theorem proved in Robert Simson's *Apollonii Pergaei Locorum Planorum Libri II.*, pp. 179-180 (1749).

† C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 7 (1825).

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(22) *That point the sum of the squares of whose distances from the vertices of a triangle is a minimum is the centroid of the triangle.**

If the triangle ABC is fixed, G is a fixed point, and AG, BG, CG fixed distances. Hence for any variable point P, $AP^2+BP^2+CP^2$ always exceeds the constant quantity $AG^2+BG^2+CG^2$ by $3PG^2$. The nearer therefore P approaches to G, the nearer does $AP^2+BP^2+CP^2$ approach this constant quantity.

(23) *That point inside a triangle which has the product of its distances from the three sides a maximum is the centroid of the triangle.*†

Let G be any point inside ABC, and GR, GS, GT its distances from BC, CA, AB.

Then $GR \times GS \times GT$ is a maximum,
when $GR \cdot \frac{1}{2}BC \times GS \cdot \frac{1}{2}CA \times GT \cdot \frac{1}{2}AB$ is a maximum,
that is, when $GBC \times GCA \times GAB$ is a maximum,
that is, when these three triangles are equal,
that is, when G is the centroid.

(24) *If three straight lines drawn from the vertices of a triangle are concurrent, the three straight lines drawn parallel to them from the mid points of the opposite sides are also concurrent; and the straight line joining the two points of concurrency passes through the centroid of the triangle and is there trisected.* ‡

The triangles ABC, A′B′C′ are similar and oppositely situated, G is their homothetic centre, and 2 : 1 is the ratio of similitude.

Hence if AD, BE, CF be concurrent at O, the corresponding straight lines A′D′, B′E′, C′F′ will pass through the corresponding point O′; OO′ the straight line joining two corresponding points, will pass through the homothetic centre G; and OG : O′G = 2 : 1.

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* J. F. de Tuschis a Fagnano in *Nova Acta Eruditorum, anni* 1775, p. 290. The article referred to is entitled: *Problemata quaedam ad methodum maximorum et minimorum spectantia*, and the volume in which it occurs was published at Leipzig in 1779.

† H. Watson in the *Ladies' Diary* for 1756.

‡ Frégier in Gergonne's *Annales*, VII. 170 (1816-7).

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(25) *If ABC be a triangle, O any point whatever, and A_1, B_1, C_1 symmetrical to O with respect to the mid points of BC, CA, AB, then* *

(a) *AA_1, BB_1, CC_1 are concurrent at a point P.*

(b) *The straight line OP turns round a fixed point G when the point O moves in any manner whatever.*

(c) *The point G divides OP in a constant ratio.*

FIGURE 9.

Let A', B', C' be the mid points of BC, CA, AB.

(*a*) Then A_1B_1 is parallel to A'B' and equal to 2A'B';
therefore it is equal and parallel to AB, but oppositely directed.
Similarly B_1C_1 and C_1A_1 are equal and parallel to BC and CA, but oppositely directed.
The three pairs of parallels BC and B_1C_1, CA and C_1A_1, AB and A_1B_1 form therefore three parallelograms, whose diagonals AA_1, BB_1, CC_1 cut each other at P the mid point of each of them.

(*b*) In triangle OAA_1 the lines OP, AA' are medians;
therefore OP cuts AA' at G such that AG = 2A'G.
But AA' is a median of triangle ABC;
therefore G is the centroid of ABC, and consequently a fixed point.

(*c*) OP is divided at G so that OG = 2GP.

(26) The sum of the squares on the sides of the complementary triangle is one-fourth of the sum of the squares on the sides of the fundamental triangle.

(27) If in a triangle its complementary triangle be inscribed, and in the complementary triangle its complementary triangle be inscribed, and so on, the limit of the sum of the squares on the sides of all the triangles so formed is one-third of the sum of the squares on the sides of the fundamental triangle.†

(28) If in a triangle its complementary triangle be inscribed, and so on, the limit to which these triangles tend is a point, and the sum of the squares on the lines drawn therefrom to the vertices of

* Mr Maurice d'Ocagne in the *Nouvelles Annales*, 3rd Series, I. 239 (1882); proof on p. 430.

† Leybourn's *Mathematical Repository*, new series, V. 111 (1820).

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all the inscribed triangles is one-third of the sum of the squares on the lines drawn from the same point to the vertices of the fundamental triangle.*

(29) *If $A_1B_1C_1$ be the complementary triangle of ABC, $A_2B_2C_2$ the complementary triangle of $A_1B_1C_1$, and so on; and if P be any point in the plane of the triangle, then*†

$$PA_n^2 + PB_n^2 + PC_n^2 = 3PG^2 + \frac{1}{3 \cdot 4^n}(BC^2 + CA^2 + AB^2).$$

FIGURE 10.

Join P with A, B, C, A′, G, and with D the mid point of AG.

Then $$AB^2 + AC^2 = 2A_1A^2 + 2A_1B^2;$$
$$= 18A_1G^2 + 2A_1B^2;$$

therefore $$BC^2 + CA^2 + AB^2 = 18A_1G^2 + 6A_1B^2.$$

Again, $$PG^2 + PA^2 = 2\,(GD^2 + PD^2),$$
$$PB^2 + PC^2 = 2\,(A_1B^2 + PA_1^2),$$
$$2\,(PA_1^2 + PD^2) = 4\,(A_1G^2 + PG^2);$$

therefore by addition, and subtraction of what is common,

$$PA^2 + PB^2 + PC^2 = 3PG^2 + 6A_1G^2 + 2A_1B^2$$
$$= 3PG^2 + \tfrac{1}{3}(BC^2 + CA^2 + AB^2).$$

Similarly $$PA_1^2 + PB_1^2 + PC_1^2 = 3PG^2 + \tfrac{1}{3}(B_1C_1^2 + C_1A_1^2 + A_1B_1^2)$$
$$= 3PG^2 + \frac{1}{3 \cdot 4}(BC^2 + CA^2 + AB^2)$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

Hence $$PA_n^2 + PB_n^2 + PC_n^2 = 3PG^2 + \frac{1}{3 \cdot 4^n}(BC^2 + CA^2 + AB^2)$$

(30) *If the sides of triangle ABC be divided at A_1, B_1 C_1, so that $BA_1 : BC = CB_1 : CA = AC_1 : AB = m$, then*

$$A_1B_1C_1 = ABC\{1 - 3m(1 - m)\}.$$

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* Mr E. Conolly in *Mathematical Questions from the Educational Times*, IV. 76 (1865).

† Mr Stephen Watson, in *Mathematical Questions from the Educational Times*, XX. 109-112 (1873), where four solutions are given. The solution in the text is Mr Watson's.

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FIGURE 11.

For $$A_1B_1C_1 = ABC - AB_1C_1 - BC_1A_1 - CA_1B_1.$$

Now $$\frac{AB_1C_1}{ABC} = \frac{AC_1}{AB} \cdot \frac{AB_1}{AC} = m(1-m);$$

therefore $$AB_1C_1 = ABC \cdot m(1-m).$$

Similarly $$BC_1A_1 = ABC \cdot m(1-m),$$

and $$CA_1B_1 = ABC \cdot m(1-m);$$

therefore $$A_1B_1C_1 = ABC\{1 - 3m(1-m)\}.$$

(31) Let there be a series of triangles

$$A_1B_1C_1,\ A_2B_2C_2,\ \ldots\ A_nB_nC_n$$

such that each is derived from the preceding in the same way as $A_1B_1C_1$ was derived from ABC; and let them be denoted by $\Delta_1, \Delta_2, \ldots \Delta_n$.

Then the formula

$$\Delta_1 = \Delta\{1 - 3m(1-m)\}$$

may be applied to triangle $A_2B_2C_2$;

therefore $$\Delta_2 = \Delta_1\{1 - 3m(1-m)\}$$
$$= \Delta\{1 - 3m(1-m)\}^2.$$

Similarly $$\Delta_3 = \Delta\{1 - 3m(1-m)\}^3,$$

and $$\Delta_n = \Delta\{1 - 3m(1-m)\}^n.$$

Hence $\Delta, \Delta_1, \Delta_2 \ldots$ form a decreasing geometrical progression, whose sum to infinity is equal to $\dfrac{\Delta}{3m(1-m)}$.

(32) This sum is a minimum when the product $m(1-m)$ is a maximum, that is, when $m = \frac{1}{2}$. Hence the minimum sum is $\frac{4}{3}\Delta$, and each triangle is then the complementary triangle of its predecessor.

When m is 0 or 1, the sum becomes infinite. This arises from the fact that then $\Delta_1, \Delta_2, \ldots$ coincide with Δ.

When m varies from 0 to 1, the sum diminishes from infinity to its maximum $\frac{4}{3}\Delta$, and then increases to infinity.

(33) *The centroid G of ABC is the centroid of* $A_1B_1C_1$, $A_2B_2C_2$, ...

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FIGURE 11.

Through B_1, C_1 draw parallels to AB, AC; these parallels will intersect on BC at a point D such that

$$BD : DC = AB_1 : B_1C = BC_1 : C_1A.$$

Hence $BD = CA_1$, and AD, B_1C_1, the diagonals of the parallelogram AB_1DC_1, bisect each other at E. Now if A′ be the mid point of BC, it will also be the mid point of DA_1;

therefore AA′, A_1E, two medians of triangle ADA_1, intersect at a point G such that

$$AG = 2A'G \text{ and } A_1G = 2EG.$$

Hence since AA′, A_1E are medians of ABC, $A_1B_1C_1$ these two triangles have the same centroid G.

What has been proved with regard to ABC, $A_1B_1C_1$ will hold equally with regard to $A_1B_1C_1$, $A_2B_2C_2$; and so on. Therefore the whole series of triangles have the same centroid.*

The last property may also be proved thus †:—

FIGURE 12.

Bisect BC and A_1B_1 at A′ and F;
join AA′, C_1F cutting each other at G;
and draw B_1D parallel to AB.

Then $BA_1 : CA_1 = CB_1 : AB_1$,
$= CD : BD$;
therefore $BA_1 = CD$;
therefore A′ is the mid point of A_1D;
therefore A′F is parallel to DB_1, and equal to $\frac{1}{2}DB_1$.

Again $B_1D : AB = CB_1 : CA$,
$= AC_1 : AB$;
therefore $B_1D = AC_1$;
therefore A′F is half of AC_1, and it is parallel to it;
therefore $AG = 2A'G$ and $C_1G = 2FG$;
therefore G is the centroid of both ABC and $A_1B_1C_1$.

* The theorem that $A_1B_1C_1$ has the same centroid as ABC will be found in Pappus's *Mathematical Collection*, VIII. 2. Chasles has some remarks on the theorem in his *Aperçu historique*, 2nd ed., p. 44.

† This mode of proof was communicated to me by Mr A. J. Pressland. Compare also Fuhrmann's *Synthetische Beweise planimetrischer Sätze*, pp. 48-9 (1890).

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Relations which Exist Between a Triangle and the Triangle whose Sides are the Medians of the Former.*

(34) *If ABC be any triangle, another triangle can always be constructed whose sides are equal to the medians of ABC.*

Figure 13.

Let AA′, BB′, CC′ be the medians of ABC.

Through A′ draw A′L parallel to BB′, and produce it so that A″L = A′L; join A″B′, A′C′.

Because A′ is the mid point of BC, and A′L is parallel to BB′, therefore L is the mid point of B′C.

Hence B′A′CA″ is a parallelogram, as well as B′BA′A″,

and $A'A'' = BB'$.

Since A B′ is equal and parallel to C′A′

and B′A″ „ „ „ „ „ A′C;

therefore A″A „ „ „ „ „ CC′;

that is AA′A″ is the triangle required.

(35) *The sides of AA′A″ are parallel to the medians of ABC*

and „ „ „ *ABC* „ „ „ „ „ „ *AA′A″.*

The first part of the theorem has been already proved.

Since $AB' : B'L = 2 : 1$

therefore B′ is the centroid of AA′A″,

Now the median B′A″ is parallel to BC,

„ „ B′A coincides with CA,

and „ „ B′A′ is parallel to AB.

(36) A″, B′, C′ are collinear.

(37) *If ABC be a triangle whose centroid is G, DEF the triangle whose sides are the medians of ABC, that is* $EF = AA'$, $FD = BB'$, $DE = CC'$, *then*

$$\angle D = GBC + GCB, \quad \angle E = GCA + GAC, \quad \angle F = GAB + GBA.$$

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* In connection with this subject, the following authorities may be consulted:

Gergonne's *Annales*, II. 93 (1811).

Supplemente zu G. S. Klügel's Wörterbuche der reinen Mathematik, Vol. I. Art. "Dreieck" (J. A. Grunert), p. 706 (1833).

Nouvelles Annales, III. 457-460 (1844).

Battaglini's *Giornale di Matematiche*, I. 126-7 (1863).

Grunert's *Archiv*, XLI. 112-4 (1864).

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FIGURE 13.

The angles which are equal have been marked with the same number; and the triangle DEF corresponds to the triangle A″AA′.

(38) *If* $ABC,\ A_1B_1C_1,\ A_2B_2C_2\ \ldots\ldots$
$DEF,\ D_1E_1F_1,\ D_2E_2F_2\ \ldots\ldots$

be two sets of triangles such that the sides of

DEF *are equal to the medians of* ABC
$A_1B_1C_1$ „ „ „ „ „ „ DEF
$D_1E_1F_1$ „ „ „ „ „ „ $A_1B_1C_1$,

and so on;
the triangles $ABC,\ A_1B_1C_1,\ \ldots$ *will be similar to each other* *
and $DEF,\ D_1E_1F_1\ \ldots$ „ „ „ „ „ „

FIGURE 14.

The proof of the theorem will appear from the figure † if it be observed that

Triangles	correspond to	
D E F	A″A A′	(4, 5; 6, 1; 2, 3),
$A_1B_1C_1$	L A‴A	(1, 2; 3, 4; 5, 6),
$D_1E_1F_1$	A M A^{IV}	(4, 5; 6, 1; 2, 3),
$A_2B_2C_2$	A^{V}A N	(1, 2; 3, 4; 5, 6),
$D_2E_2F_2$	P A^{VI}A	(4, 5; 6, 1; 2, 3).

The theorem may be proved also as follows:‡

If $m_1,\ m_2\ m_3$, be the three medians of ABC, then

$$m_1^2 = \tfrac{1}{2}(b^2 + c^2 - \tfrac{1}{2}a^2),\quad m_2^2 = \tfrac{1}{2}(c^2 + a^2 - \tfrac{1}{2}b^2),\quad m_3^2 = \tfrac{1}{2}(a^2 + b^2 - \tfrac{1}{2}c^2).$$

Make a triangle whose sides are $m_1,\ m_2,\ m_3$,
and let its medians be $a_1,\ b_1,\ c_1$; then

$$a_1^2 = \tfrac{1}{2}(m_2^2 + m_3^2 - \tfrac{1}{2}m_1^2), b_1^2 = \tfrac{1}{2}(m_3^2 + m_1^2 - \tfrac{1}{2}m_2^2), c_1^2 = \tfrac{1}{2}(m_1^2 + m_2^2 - \tfrac{1}{2}m_3^2)$$
$$= \tfrac{9}{16}a^2 \quad , \quad = \tfrac{9}{16}b^2 \quad , \quad = \tfrac{9}{16}c^2;$$

therefore $a_1 = \tfrac{3}{4}a,\ b_1 = \tfrac{3}{4}b,\ c_1 = \tfrac{3}{4}c,$

and $a_1 : b_1 : c_1 = a : b : c.$

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* Gergonne's *Annales*, II. 93 (1811).

† The figure has been taken from Grunert's article "Dreieck" previously referred to.

‡ Grunert's *Archiv*, XLI. 112-4 (1864).

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(39) If $\Delta, \Delta_2, \Delta_4 \ldots\ldots$

$\Delta_1, \Delta_3, \Delta_5 \ldots\ldots$

denote the two sets of triangles in (38), the sides of

Δ	are	a,	b,	c
Δ_2	,,	$\frac{3}{4}a$,	$\frac{3}{4}b$,	$\frac{3}{4}c$
Δ_4	,,	$(\frac{3}{4})^2a$,	$(\frac{3}{4})^2b$,	$(\frac{3}{4})^2c$
Δ_{2n}	,,	$(\frac{3}{4})^na$,	$(\frac{3}{4})^nb$,	$(\frac{3}{4})^nc$
Δ_1	,,	m_1,	m_2,	m_3
Δ_3	,,	$\frac{3}{4}m_1$,	$\frac{3}{4}m_2$,	$\frac{3}{4}m_3$
Δ_5	,,	$(\frac{3}{4})^2m_1$,	$(\frac{3}{4})^2m_2$,	$(\frac{3}{4})^2m_3$
Δ_{2n+1}	,,	$(\frac{3}{4})^nm_1$,	$(\frac{3}{4})^nm_2$,	$(\frac{3}{4})^nm_3$

(40) *The triangles* $\Delta, \Delta_1, \Delta_2, \Delta_3 \ldots$ *form a geometrical progression* * *whose common ratio is* $\frac{3}{4}$.

FIGURE 13.

Since $AL = \frac{3}{4}AC$

therefore $AA'L = \frac{3}{4}AA'C$;

therefore $AA'A'' = \frac{3}{4}AB\,C$; and so on.

(41) $\Delta + \Delta_1 + \Delta_2 + \ldots ad\ infinitum = 4\Delta$.

(42) If $p, p_2, p_4 \ldots$ be the perimeters of $\Delta, \Delta_2, \Delta_4 \ldots$

$p + p_2 + p_4 + \ldots ad\ infinitum = 4p$.

(43) If $p_1, p_3, p_5 \ldots$ be the perimeters of $\Delta_1, \Delta_3, \Delta_5 \ldots$

$p_1 + p_3 + p_5 + \ldots ad\ infinitum = 4p_1$.

(44) $\Delta + \Delta_2 + \Delta_4 + \ldots ad\ infinitum = \frac{16}{7}\Delta$.

(45) $\Delta_1 + \Delta_3 + \Delta_5 + \ldots ad\ infinitum = \frac{12}{7}\Delta$.

(46) If G be the centroid of ABC and another triangle $A_0B_0C_0$ be formed with sides respectively equal to $\sqrt{3}\,GA$, $\sqrt{3}\,GB$, $\sqrt{3}\,GC$, then ABC may be derived from $A_0B_0C_0$ in the same way as the latter was derived from the former, that is, the relation between the triangles is a conjugate one.†

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* Gergonne's *Annales*, II. 93 (1811).

† Rev. T. C. Simmons in Milne's *Companion to the Weekly Problem Papers*, pp. 150-1 (1888).

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(47) The areas of ABC, $A_0B_0C_0$ are equal.*

These two theorems follow without much difficulty from what precedes.

(48) *If through the centroid G of a triangle ABC a straight line be drawn cutting BC, CA, AB in D, E, F and the points E, F be on the same side of G then*†

$$\frac{1}{GE}+\frac{1}{GF}=\frac{1}{GD}.$$

FIGURE 15.

Through A draw AN parallel to BC meeting DEF in K, and through G draw LMN parallel to AB meeting BC, CA, AN in L, M, N.

Then $LG = MG$, and $CM = 2AM$.

But since triangles CML, AMN are similar,

therefore $ML = 2MN$;

therefore $GM = MN$.

Hence AG, AM, AN, AF form a harmonic pencil;

and they are cut by the transversal GEKF;

therefore G, E, K, F form a harmonic range;

therefore
$$\frac{1}{GE}+\frac{1}{GF}=\frac{2}{GK}$$
$$=\frac{1}{GD}$$

since $GK = 2GD$.

---

* Rev. T. C. Simmons in Milne's *Companion to the Weekly Problem Papers*, p. 151 (1888).

† This property, proved in the manner given, will be found in Maclaurin's *Algebra* (1748) in the Appendix, *De Linearum Geometricarum Proprietatibus generalibus Tractatus*, §98 or p. 57. A proof by Dr E. v. Hunyady of Pesth, by means of transversals, will be found in Schlömilch's *Zeitschrift*, VII. 268-9 (1862).

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Formulæ connected with the Medians.

The medians in terms of the sides.

$$\left.\begin{aligned} 4m_1^2 &= -a^2+2b^2+2c^2 \\ 4m_2^2 &= 2a^2-b^2+2c^2 \\ 4m_3^2 &= 2a^2+2b^2-c^2 \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} 4m_1^2 &= -a^2+(b+c)^2+(b-c)^2 \\ 4m_2^2 &= -b^2+(c+a)^2+(c-a)^2 \\ 4m_3^2 &= -c^2+(a+b)^2+(a-b)^2 \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} (2m_1+b-c)(2m_1-b+c) &= (a+b+c)(-a+b+c) \\ (2m_2+c-a)(2m_2-c+a) &= (a+b+c)(a-b+c) \\ (2m_3+a-b)(2m_3-a+b) &= (a+b+c)(a+b-c) \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} (b+c+2m_1)(b+c-2m_1) &= (a+b-c)(a-b+c) \\ (c+a+2m_2)(c+a-2m_2) &= (-a+b+c)(a+b-c) \\ (a+b+2m_3)(a+b-2m_3) &= (a-b+c)(-a+b+c) \end{aligned}\right\} \quad (4)$$

$$4(m_1^2+m_2^2+m_3^2) = 3(a^2+b^2+c^2) \quad (5)$$

$$3(AG^2+BG^2+CG^2) = a^2+b^2+c^2 \quad (6)$$

$$12(A'G^2+B'G^2+C'G^2) = a^2+b^2+c^2 \quad (7)$$

$$m_1 \cdot AG + m_2 \cdot BG + m_3 \cdot CG = \tfrac{1}{2}(a^2+b^2+c^2) \quad (8)$$

$$m_1 \cdot A'G + m_2 \cdot B'G + m_3 \cdot C'G = \tfrac{1}{4}(a^2+b^2+c^2) \quad (9)$$

$$16(m_1^4+m_2^4+m_3^4) = 9(a^4+b^4+c^4) \quad (10)$$

$$16(m_2^2m_3^2+m_3^2m_1^2+m_1^2m_2^2) = 9(b^2c^2+c^2a^2+a^2b^2) \quad (11)$$

The sides in terms of the medians.

$$\left.\begin{aligned} 9a^2 &= -4m_1^2+8m_2^2+8m_3^2 \\ 9b^2 &= 8m_1^2-4m_2^2+8m_3^2 \\ 9c^2 &= 8m_1^2+8m_2^2-4m_3^2 \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} \tfrac{9}{4}a^2 &= -m_1^2+(m_2+m_3)^2+(m_2-m_3)^2 \\ \tfrac{9}{4}b^2 &= -m_2^2+(m_3+m_1)^2+(m_3-m_1)^2 \\ \tfrac{9}{4}c^2 &= -m_3^2+(m_1+m_2)^2+(m_1-m_2)^2 \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned} (\tfrac{3}{2}a+m_2-m_3)(\tfrac{3}{2}a-m_2+m_3) &= (m_1+m_2+m_3)(-m_1+m_2+m_3) \\ (\tfrac{3}{2}b+m_3-m_1)(\tfrac{3}{2}b-m_3+m_1) &= (m_1+m_2+m_3)(m_1-m_2+m_3) \\ (\tfrac{3}{2}c+m_1-m_2)(\tfrac{3}{2}c-m_1+m_2) &= (m_1+m_2+m_3)(m_1+m_2-m_3) \end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned} (m_2+m_3+\tfrac{3}{2}a)(m_2+m-\tfrac{3}{2}a) &= (m_1+m_2+m_3)(m_1-m_2+m_3) \\ (m_3+m_1+\tfrac{3}{2}b)(m_3+m_1-\tfrac{3}{2}b) &= (m_1+m_2+m_3)(m_1+m_2-m_3) \\ (m_1+m_2+\tfrac{3}{2}c)(m_1+m_2-\tfrac{3}{2}c) &= (-m_1-m_2+m_3)(-m_1+m_2+m_3) \end{aligned}\right\} \quad (15)$$

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$$\left.\begin{aligned} 3(b^2 \sim c^2) &= 4(m_3^2 \sim m_2^2) \\ 3(c^2 \sim a^2) &= 4(m_1^2 \sim m_3^2) \\ 3(a^2 \sim b^2) &= 4(m_2^2 \sim m_1^2) \end{aligned}\right\} \quad (16)$$

If

$$\begin{aligned} 2s &= a+b+c \\ 2s' &= b+c+2m_1 \\ 2s'' &= c+a+2m_2 \\ 2s''' &= a+b+2m_3 \end{aligned}$$

then (3) and (4) become

$$\left.\begin{aligned} (s'-b)(s'-c) &= s(s-a) & s'(s'-2m_1) &= (s-b)(s-c) \\ (s''-c)(s''-a) &= s(s-b) & s''(s''-2m_2) &= (s-c)(s-a) \\ (s'''-a)(s'''-b) &= s(s-c) & s'''(s'''-2m_3) &= (s-a)(s-b) \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned} \Delta^2 &= s'(s'-b)(s'-c)(s'-2m_1) \\ &= s''(s''-c)(s''-a)(s''-2m_2) \\ &= s'''(s'''-a)(s'''-b)(s'''-2m_3) \end{aligned}\right\} \quad (18)$$

For each $= s(s-a)(s-b)(s-c)$

If

$$\begin{aligned} 2m &= m_1+m_2+m_3 \\ 2n_1 &= m_2+m_3+\tfrac{3}{2}a \\ 2n_2 &= m_3+m_1+\tfrac{3}{2}b \\ 2n_3 &= m_1+m_2+\tfrac{3}{2}c \end{aligned}$$

then (14) and (15) become

$$\left.\begin{aligned} (n_1-m_2)(n_1-m_3) &= m(m-m_1) & n_1(n_1-\tfrac{3}{2}a) &= (m-m_2)(m-m_3) \\ (n_2-m_3)(n_2-m_1) &= m(m-m_3) & n_2(n_2-\tfrac{3}{2}b) &= (m-m_3)(m-m_1) \\ (n_3-m_1)(n_3-m_2) &= m(m-m_3) & n_3(n_3-\tfrac{3}{2}c) &= (m-m_1)(m-m_2) \end{aligned}\right\} \quad (19)$$

Area of triangle in terms of its medians.

FIGURE 1.

Because $\mathrm{ABC} = 2\mathrm{ABA}' = 6\mathrm{GBA}' = 3\mathrm{GBL}$;

and $\mathrm{GL} = \tfrac{2}{3}m_1$, $\mathrm{GB} = \tfrac{2}{3}m_2$, $\mathrm{BL} = \tfrac{2}{3}m_3$;

therefore $\Delta^2 = 9(\mathrm{GBL})^2$

$$= 9\left\{\frac{m_1+m_2+m_3}{3}\cdot\frac{-m_1+m_2+m_3}{3}\cdot\frac{m_1-m_2+m_3}{3}\cdot\frac{m_1+m_2-m_3}{3}\right\}$$

$$= \tfrac{1}{9}\{(m_1+m_2+m_3)(-m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)\}.$$

Let $2m = m_1+m_2+m_3$

then $\Delta^2 = \tfrac{16}{9}\{m(m-m_1)(m-m_2)(m-m_3)\}$ (20)

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If
$$\begin{aligned} 2m' &= -m_1 + m_2 + m_3 \\ 2m'' &= m_1 - m_2 + m_3 \\ 2m''' &= m_1 + m_2 - m_3 \end{aligned}$$
then
$$\triangle = \tfrac{4}{3}\sqrt{mm'm''m'''} \tag{21}$$

$$\left.\begin{aligned} \triangle &= \tfrac{4}{3}\sqrt{n_1(n_1-m_2)(n_1-m_3)(n_1-\tfrac{3}{2}a)} \\ &= \tfrac{4}{3}\sqrt{n_2(n_2-m_3)(n_2-m_1)(n_2-\tfrac{3}{2}b)} \\ &= \tfrac{4}{3}\sqrt{n_3(n_3-m_1)(n_3-m_2)(n_3-\tfrac{3}{2}c)} \end{aligned}\right\} \tag{22}$$

This is deduced from (20) by means of (19).

If R, S, T be the projections of G on the sides

$$\left.\begin{aligned} \mathrm{BR} &= \frac{3a^2 - b^2 + c^2}{6a} & \mathrm{CR} &= \frac{3a^2 + b^2 - c^2}{6a} \\ \mathrm{CS} &= \frac{a^2 + 3b^2 - c^2}{6b} & \mathrm{AS} &= \frac{-a^2 + 3b^2 + c^2}{6b} \\ \mathrm{AT} &= \frac{-a^2 + b^2 + 3c^2}{6c} & \mathrm{BT} &= \frac{a^2 - b^2 + 3c^2}{6c} \end{aligned}\right\} \tag{23}$$

$$\mathrm{ST} = \frac{4m_1\triangle}{3bc} \qquad \mathrm{TR} = \frac{4m_2\triangle}{3ca} \qquad \mathrm{RS} = \frac{4m_3\triangle}{3ab} \tag{24}$$

$$\mathrm{ST} : \mathrm{TR} : \mathrm{RS} = am_1 : bm_2 : cm_3 \tag{25}$$

Of the preceding formulæ, (8) and (9) are given by C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 7 (1825); (10) and (11) occur in Hind's *Trigonometry*, 4th ed., p. 244 (1841); (12) in Thomas Simpson's *Select Exercises*, Part II., Problem xxii. (1752); (2)-(4), (13)-(19), (21), (22) are due to Thomas Weddle. See *Lady's and Gentleman's Diary* for 1848, pp. 74-75. I have changed the notation adopted by Weddle.

On the authority of Férussac's *Bulletin des Sciences Mathématiques*, xii. 297 (1829), formula (20) should be assigned to Professor Desgranges.

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§ 2. Circumcentre.

*The perpendiculars to the sides of a triangle from the mid points of the sides are concurrent.**

The following demonstration† may be compared with the demonstration of § 5.

Figure 16.

Let A′, B′, C′ be the mid points of BC, CA, AB,
and let A′X′, B′Y′, C′Z′ be perpendicular to BC, CA, AB.

Join B′C′, C′A′, A′B′.

Then B′C′ is parallel to BC;
therefore A′X′ is perpendicular to B′C′.
Hence B′Y′ „ „ „ C′A′
and C′Z′ „ „ „ A′B′.

If therefore it be assumed as true that the perpendiculars to the sides of a triangle from the opposite vertices are concurrent,

A′X′, B′Y′, C′Z′ are concurrent.

Another demonstration is obtained at once from the theorem, ‡

If three points be taken on the sides of a triangle such that the sums of the squares of the alternate segments taken cyclically are equal, the perpendiculars to the sides of the triangle at these points are concurrent.

The point of concurrency, which will be denoted by O, is the centre of the circle circumscribed about ABC. This circle is often called the *circumcircle*,§ and the centre of it the *circumcentre*.§

The radius of the circumcircle is usually denoted by R.

(1) The circumcentre of a triangle is the orthocentre of its complementary triangle.

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* Euclid's *Elements*, IV. 5.

† C. Adams, *Die Lehre von den Transversalen*, p. 21 (1843).

‡ F. G. de Oppel, *Analysis Triangulorum*, p. 32 (1746).

§ These terms as well as *incircle, excircle, midcircle, incentre, excentre, midcentre* were suggested by W. H. H. H[udson]. See *Nature*, XXVIII. 7, 104 (1883). The terms *Umkreis, Inkreis, Ankreis, Mittenkreis* have been more or less in use in Germany since 1866, as may be seen from Schlömilch's *Zeitschrift*.

The perpendiculars to the sides of a triangle from the mid points of the sides are sometimes called *médiatrices* in France and Belgium.

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(2) Since the complementary and the fundamental triangles are similar, and since their sides are in the ratio of 1 to 2, the distance of the circumcentre of a triangle from any side is half of the distance between the orthocentre of the triangle and the vertex opposite that side.*

(3) If O be the circumcentre of a triangle ABC, the circle on OA as diameter bisects AB and AC.

Similarly for the circles on OB and OC.

(4) *If the circle on OA as diameter should cut BC at P and P', then AP is a mean proportional† between BP and CP, and AP' is a mean proportional between BP' and CP'.*

FIGURE 17.

Produce AP to meet the circumcircle at Q.

The circle on OA as diameter touches the circumcircle at A; therefore A is the homothetic centre of the two circles;

therefore $AP : AQ = 1 : 2$;

therefore $BP \cdot CP = AP \cdot QP = AP^2$.

(5) *By the following construction‡ a point P will be found in the base BC of ABC such that the ratio $AP^2 : BP \cdot CP$ has a given value.*

FIGURE 18.

Join AO, and divide it at L so that AL : LO has the given value; then the circle with centre L and radius LA will meet BC in two points P, P' satisfying the condition.

Produce AP to meet the circumcircle in Q.

Then
$$\begin{aligned} AP : PQ &= AP^2 : AP \cdot PQ \\ &= AP^2 : BP \cdot PC \\ &= AL : LO \quad ; \end{aligned}$$

therefore LP is parallel to OQ;
therefore LP = LA, since OQ = OA.

---

* *Ladies' Diary* for 1785.

† Given without proof in the *Ladies' Diary* for 1759.

‡ Rev. J. Wolstenholme in the *Educational Times*, XXIX., 273 (1877). Four solutions are given in *Mathematical Questions from the Educational Times*, XXVII, 63-66 (1877); the one in the text is the last.

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If AP'Q' be the other position of APQ,

then $AP : PQ = AP' : P'Q'$;

therefore QQ' is parallel to BC;

therefore arc BQ = arc CQ',

and AP, AP' are isogonal with respect to $\angle A$.

(6) If from a point O within or without a triangle ABC, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB, and circles are circumscribed about the triangles OEF, OFD, ODE; the area of the triangle formed by joining the centres of these circles is one-fourth of the area * of the triangle ABC

FIGURE 19.

The centres of these circles are the mid points of OA, OB, OC.

(7) *If from a point O within triangle ABC perpendiculars OD, OE, OF be drawn to BC, CA, AB, then*†

$$2R(EF + FD + DE) = OA \cdot BC + OB \cdot CA + OC \cdot AB.$$

FIGURE 19.

For A, F, O, E lie on the circle whose diameter is OA, and the chord EF subtends the same angle A at the circumference of this circle as BC does at that of the circumcircle of ABC;

therefore $EF : OA = BC : 2R$;

therefore $2R \cdot EF = OA \cdot BC$.

(8) If O be on that arc of the circumcircle on which angle C stands, then,† by Ptolemy's theorem,

$$OA \cdot BC + OB \cdot CA - OC \cdot AB = 0;$$

therefore $EF + FD - DE = 0$;

therefore D, E, F are collinear,

which is another proof of the property of the Wallace line.

---

* Todhunter's *Plane Trigonometry*, Chap. XVI., Ex. 41 (1859).

† Both (7) and (8) are due to Mr E. M. Langley, who applies the first of them to the problem of finding the triangle of minimum perimeter inscribed in a given triangle, and to the determination of the trilinear co-ordinates of the Brocard points. See *Sixteenth General Report of the Association for the Improvement of Geometrical Teaching*, pp. 34-5 (1890).

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(9) *If O be the circumcentre of ABC, and AO, BO, CO be produced to meet the circumcircle in U, V, W, the triangle UVW is congruent to ABC.*

FIGURE 20.

For $\angle AUV = \angle ABO = \angle BAO$,

$\angle AUW = \angle ACO = \angle CAO$;

therefore $\angle VUW = \angle BAC$;

therefore UVW is similar to ABC.

But these two triangles are inscribed in the same circle;

therefore they are congruent.

(10) The figures BCVW, CAWU, ABUV are rectangles.

(11) If ABC be a triangle, and BW, CV be perpendicular to BC; CU, AW perpendicular to CA; AV, BU perpendicular to AB, the three straight lines AU, BV, CW are concurrent at the circumcentre of ABC, and the six points A, B, C, U, V, W are concyclic.*

FIGURE 20.

(12) *Triangles $A_1B_1C_1$, $A_2B_2C_2$ circumscribed about ABC in such a manner that their sides are perpendicular to those of ABC are congruent † to each other and similar to ABC.*

FIGURE 21.

Let C_1A_1, A_2B_2; A_1B_1, B_2C_2; B_1C_1, C_2A_2

meet at U, V, W.

Then BB_1VB_2 is a parallelogram;

therefore $B B_1 = B_2V$.

But $B W = C V$;

therefore $B_1W = C B_2$.

Again WC_1CC_2 is a parallelogram;

therefore $WC_1 = C_2C$;

therefore $B_1C_1 = B_1C_2$.

Similarly $C_1A_1 = C_2A_2$, $A_1B_1 = A_2B_2$.

---

* C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 56 (1825).

† The first part of the theorem is given by Jacobi, p. 56.

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Lastly $\quad \angle BAC = 90° - \angle BAA$

$= \angle B_1AA_2 = \angle A.$

Similarly $\quad \angle ABC = \angle B_1, \ \angle ACB = C_1.$

This theorem is a particular case of a more general one.

(13) *The three lines A_1A_2, B_1B_2, C_1C_2 are concurrent* at the circumcentre of ABC.*

FIGURE 21.

For AU, BV, CW are concurrent at O, the circumcentre of ABC; and O is the mid point of AU, BV, CW.

Now since AA_1UA_2 is a parallelogram,

therefore A_1A_2 passes through the mid point of AU.

Similarly for B_1B_2, C_1C_2.

(14) *If a point P be taken inside the triangle ABC, and circles be circumscribed about the triangles PBC, PCA, PAB, and their centres O_1, O_2, O_3 be joined, the angles of triangle $O_1O_2O_3$ are supplementary to the angles BPC, CPA, APB.*

FIGURE 22.

For O_2O_3, O_3O_1, O_1O_2 are respectively perpendicular to PA, PB, PC.

(15) *If through A any straight line MN be drawn meeting the circumferences PCA, PAB in M, N, then MC, NB will intersect on the circumference† PBC.*

Let MC, NB intersect at L.

Then $\quad \angle M = 180° - \angle CPA,$

$\angle N = 180° - \angle APB;$

therefore $\quad \angle M + \angle N = 360° - (\angle CPA + \angle APB),$

$= \angle BPC;$

therefore $\quad \angle L = 180° - \angle BPC;$

therefore L is on the circumference PBC.

---

* Jacobi does not state this property, but from the way in which he letters the figures of theorems (11) and (12) it is probable that he knew it. The property is explicitly stated, along with some others, by Mr Lemoine in his paper read at the Lyons meeting (1873) of the *Association Française pour l'avancement des Sciences.*

† Rochat in Gergonne's *Annales*, II. 29 (1811). To him also are due (17) and (19).

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(16) If L be any point on the circumference PBC, and if LC, LB meet the circumferences PCA, PAB again in M, N, then M, A, N are collinear.

(17) Triangle LMN is similar to $O_1O_2O_3$.

If the point P be fixed, the triangles $O_1O_2O_3$, LMN are given in species.

(18) *The angles which MN, NL, LM make with AP, BP, CP respectively are equal.*

For $$\angle PAN = 180° - \angle PBN = \angle PBL$$
$$= 180° - \angle PCL = \angle PCM.$$

(19) *Of all the triangles such as LMN whose sides pass through A, B, C, and whose vertices are situated on the circles O_1, O_2, O_3, that triangle L'M'N' is a maximum whose sides are perpendicular to AP, BP, CP.*

FIGURE 22.

For triangles L'M'N', LMN are similar,

and PL', PL are corresponding lines in these triangles.

Now PL' is a diameter of the circle O_1;

therefore PL' is greater than PL;

therefore L'M'N' is greater than LMN.

(20) If O be the circumcentre of ABC, and about the triangles OBC, OCA, OAB circles be circumscribed whose centres are O_1, O_2, O_3, the triangle $O_1O_2O_3$ has its angles equal to $180° - 2A$, $180° - 2B$, $180° - 2C$.

It will be found that $O_1O_2O_3$ is similar to XYZ. See § 5.

(21) *O is the incentre of the triangle $O_1O_2O_3$.*

FIGURE 22.

In the diagram suppose P to be replaced by O, and let V, W be the mid points of BO, CO.

Then the right-angled triangles OVO_1, OWO_1 have two sides of the one equal to two sides of the other;

therefore OO_1 bisects $\angle O_2O_1O_3$.

Similarly for OO_2, OO_3.

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(22.) *If OO_1, OO_2, OO_3 be produced to meet the circles OBC, OCA, OAB in L', M', N', the triangle L'M'N' will be circumscribed about ABC, will be similar and similarly situated to $O_1O_2O_3$, and will have O for its incentre.*

FIGURE 22.

For $\angle OAM' + \angle OAN' = 180°$;

therefore M', A, N' are collinear, and M'N' is parallel to O_2O_3.

Since OA, OB, OC are equal, and perpendicular to M'N', N'L', L'M';

therefore O is the incentre of L'M'N'.

Many relations between the triangles $O_1O_2O_3$ and ABC may be derived from the relations between XYZ and ABC, seeing that $O_1O_2O_3$ is similar to XYZ and that the ratio of the radii of their incircles is $\frac{1}{2}R : \rho$.

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§ 3. Incentre.

*The internal angular bisectors of a triangle are concurrent.**

The following demonstration † is different from the usual one.

Figure 23.

Let AL be the internal bisector of $\angle A$, and let the internal bisector of $\angle B$ cut it at I.

Then
$$AI : LI = BA : BL$$
$$= CA : CL;$$

therefore the internal bisector of $\angle C$ passes through I.

The point of concurrency, which will be denoted by I, is the centre of the circle inscribed in ABC. This circle is often called the *incircle*,‡ and the centre of it the *incentre*.‡

The radius of the incircle is usually denoted by r.

The following method§ of inscribing a circle in a given triangle will be better understood after a perusal of § 4 (5). As regards practical execution it is the simplest yet obtained.

Figure 24.

Along CA take AP equal to AB, and CQ equal to CB.

With A as centre and PQ as radius describe a circle cutting AC, AB in the points S, T.

With S as centre and PQ as radius describe a circle cutting the first circle in two points; the straight line joining these two points passes through the incentre.

With T as centre and PQ as radius describe a circle cutting the first circle in two points; the straight line joining these two points passes through the incentre.

Hence the incentre is determined as well as the radius of the incircle.

The proof will be evident from the following considerations.

---

* Euclid's *Elements*, IV. 4.

† Todhunter's *Elements of Euclid*, p. 312 (1864).

‡ See the note on p. 32.

§ Mr E. Lemoine in the Report (second part) of the 21st session of the *Association Française pour l'avancement des sciences*, p. 49 (1892).

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The dotted straight lines bisect AS, AT perpendicularly.

Now $AS = AT = PQ = CP - CQ = b + c - a$;

therefore if E, F be the mid points of AS, AT

$$AE = AF = \tfrac{1}{2}(b + c - a) = s_1$$

and E, F are points of contact of the incircle.

(1) The area of a triangle is equal to the rectangle * under the semiperimeter † of the triangle and the radius of the incircle.

This is expressed, $\Delta = sr$,

where $s = \tfrac{1}{2}(a + b + c)$.

(2) *If P be any point inside ABC, and PA, PB, PC be denoted by* α, β, γ, *and the radii of the incircles of PBC, PCA, PAB by* ρ_1, ρ_2, ρ_3, *then*

$$(\rho_2 + \rho_3)\alpha + (\rho_3 + \rho_1)\beta + (\rho_1 + \rho_2)\gamma = (r - \rho_1)a + (r - \rho_2)b + (r - \rho_3)c.$$

For
$$2ABC = r(a + b + c),$$
$$2PBC = \rho_1(a + \beta + \gamma),$$
$$2PCA = \rho_2(b + \gamma + \alpha),$$
$$2PAB = \rho_3(c + \alpha + \beta).$$

But $ABC = PBC + PCA + PAB$;

therefore
$$\rho_1(a + \beta + \gamma) + \rho_2(b + \gamma + \alpha) + \rho_3(c + \alpha + \beta) = r(a + b + c);$$

whence the result follows.

If P be outside ABC,

$$(\rho_2 + \rho_3)\alpha + (\rho_3 - \rho_1)\beta + (-\rho_1 + \rho_2)\gamma = (r + \rho_1)a + (r - \rho_2)b + (r - \rho_3)c.$$

(3) *If I be the incentre of ABC, and AI, BI, CI be produced to meet the circumcircle in U, V, W, then the sides of UVW are perpendicular to AI, BI, CI.*

This is established in the course of the proof of Heron's theorem regarding the area of a triangle. See § 8.

† The semiperimeter of a triangle is usually denoted, in this country and North America, by s; on the continent of Europe it is generally denoted by p. Euler, who was one of the first if not the first to introduce the notation a, b, c for the sides of ABC, denotes the semiperimeter $\frac{1}{2}(AB+BC+CA)$ by S. See an article by him entitled *Variæ demonstrationes geometriæ* printed in *Novi Commentarii Academiæ Scientiarum Imperialis Petropolitanæ* for the years 1747-8, I. 53 (1750).

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FIGURE 25.

Join U, V, W with A, B, C.

The arc $BU = CU$, $CV = AV$, $AW = BW$;

therefore arc UBW = arc CU + arc AW;

therefore $\angle UCI = \angle UIC$;

therefore $UI = UC = UB$.

Similarly $VI = VC = VA$,

and $WI = WA = WB$.

Hence $AWIV$, $BUIW$, $CVIU$ are kites;

therefore VW, WU, UV are perpendicular to AI, BI, CI.

(4) *The angles of UVW are respectively equal to*

$$\tfrac{1}{2}(B+C),\ \tfrac{1}{2}(C+A),\ \tfrac{1}{2}(A+B).$$

For
$$\begin{aligned}\angle WUV &= \angle AUV + \angle AUW \\ &= \angle ABV + \angle ACW \\ &= \tfrac{1}{2}(B+C).\end{aligned}$$

Hence whatever be the size of the angles A, B, C, triangle UVW is always acute-angled.

(5) The angles of ABC expressed in terms of the angles of UVW are

$$\begin{aligned}A &= -U+V+W = 180^\circ - 2U \\ B &= U-V+W = 180^\circ - 2V \\ C &= U+V-W = 180^\circ - 2W.\end{aligned}$$

Compare § 5, (8).

(6) $$UVW : ABC = R : 2r.$$

Join the circumcentre O with A, B, C.

Then
$$\begin{aligned}2UVW &= \text{hexagon } AWBUCV \\ &= OBUC + OCVA + OAWB \\ &= \tfrac{1}{2}(OU \cdot BC + OV \cdot CA + OW \cdot AB) \\ &= \tfrac{1}{2}R \cdot 2s = Rs;\end{aligned}$$

and $2ABC = 2rs$.

(7) *If ABC, $A_1B_1C_1$, $A_2B_2C_2$, ... $A_nB_nC_n$ be a series of triangles all inscribed in the same circle and each of which is derived from the preceding in the same manner as UVW was derived from ABC in (3); then when the whole number m increases indefinitely, the*

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*triangle $A_{2m}B_{2m}C_{2m}$ tends towards a limiting position $\alpha\beta\gamma$, the triangle $A_{2m+1}B_{2m+1}C_{2m+1}$ tends also towards a limiting position $\alpha'\beta'\gamma'$, the two limiting triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are equilateral, and symmetrically placed with reference to the centre of the circle.**

FIGURE 26.

In triangle $A_1B_1C_1$, the perpendicular from A_1 to the opposite side is A_1A, the diameter of the circumcircle is A_1OD;

therefore the bisector of $\angle B_1A_1C_1$ is also † the bisector of $\angle AA_1D$;

therefore the vertex A_2 is at the middle of the arc AD intercepted by $\angle OA_1A$.

Hence in general, to obtain the vertex A_{n+1} draw the diameter OA_n and the straight line $A_{n-1}A_n$; the mid point of the arc intercepted by the inscribed angle thus formed is the vertex sought.

In this manner, step by step, the vertices A_2, A_3, A_4 ... are determined, and each time the inscribed angle diminishes by half. This angle therefore tends to become zero, and the two lines $A_{n+1}A_n$ and $A_{n-1}A_n$ end by coalescing with the diameter OA_n. Now since the first of these two lines is an angular bisector and the second is the corresponding perpendicular of the triangle $A_nB_nC_n$, this triangle tends to become isosceles, that is, in the limit, $\angle B_n = \angle C_n$.

Similarly $\angle C_n = \angle A_n$ and $\angle A_n = \angle B_n$;

hence the triangle $A_nB_nC_n$ tends to become equilateral.

The inscribed angles which give the vertices of even order are quartered each time. Hence the halves AA_2, A_2A_4, ... of the arcs intercepted by these angles form the terms of a geometrical series whose ratio is $\frac{1}{4}$. Since none of the arcs AA_2, A_2A_4, ... $A_{2m-1}A_{2m}$ encroaches on the preceding, their sum represents the distance AA_{2m}. This sum is, in the limit,

$$A\alpha = \tfrac{4}{3}AA_2.$$

Thus the position of the limiting equilateral triangle $\alpha\beta\gamma$ is known.

---

* Both (7) and (8) were proposed at a competitive examination in France in 1881. For the proofs see Vuibert's *Journal*, VII. 121-3 (1883).

† See § 5, (34).

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In the same way the triangles of odd order tend towards the equilateral triangle $\alpha'\beta'\gamma'$ which is such that

$$A_1\alpha' = \tfrac{4}{3}A_1A_3.$$

To prove that the equilateral triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are symmetrical with respect to the centre O, it is sufficient to prove that α and α' are diametrically opposite, or that $D\alpha = A_1\alpha'$.

Now
$$D\alpha = 2A\,A_2 - A\alpha = \tfrac{2}{3}AA_2,$$
$$A_1\alpha' = \tfrac{4}{3}A_1A_3 \quad = \tfrac{2}{3}AA_2.$$

(8) *If the radius R be taken as unity, the product of the numbers which measure the diameters of the circles inscribed in the triangles ABC, $A_1B_1C_1$ ... $A_nB_nC_n$ tends towards a limit when n increases indefinitely.*

Let Δ, Δ_1 ... Δ_n denote the areas of the triangles ABC, $A_1B_1C_1$ ... $A_nB_nC_n$; and d, d_1, ... d_n the numbers which measure the diameters of their incircles.

Then
$$\Delta_1 : \Delta = R : 2r = 1 : d;$$

therefore
$$d = \frac{\Delta}{\Delta_1}$$

Similarly
$$d_1 = \frac{\Delta_1}{\Delta_2} \quad d_2 = \frac{\Delta_2}{\Delta_3}, \ \ldots \ d_n = \frac{\Delta_n}{\Delta_{n+1}};$$

therefore
$$dd_1d_2 \ldots d_n = \frac{\Delta}{\Delta_{n+1}}.$$

Now when n increases indefinitely, Δ_{n+1} approaches the area of the equilateral triangle inscribed in a circle of radius 1, that is $3\sqrt{3}/4$; hence $dd_1d_2 \ldots d_n$ approaches the limit $4\Delta/3\sqrt{3}$.

(9) It has been seen that the series of triangles $A_1B_1C_1$, $A_2B_2C_2$, etc., deduced successively from ABC and from each other can be continued indefinitely far. Can this series be extended backwards indefinitely far, and if not, when will it stop? To answer the question a solution must be found for the problem:

Given a triangle ABC inscribed in a circle, construct another inscribed triangle RST such that A, B, C shall be the mid points of the arcs ST, TR, RS.

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From § 3, (4) it appears that whatever be the size of the angles R, S, T, the triangle ABC must be acute-angled. This being granted, draw the perpendiculars AX, BY, CZ of ABC, and produce them to meet the circumcircle in R, S, T. These are the vertices of the triangle sought.

The demonstration follows from the fact that H is the incentre of triangle RST. See § 5 (15).

By operating in a similar manner on RST, etc., the series may be continued backwards. It is plain, however, that as soon as a triangle is reached which is not acute-angled, the process comes to an end.

It may happen that a triangle is reached which has one angle right. Let RST be this triangle, R the right angle.

Draw RR_1 perpendicular to ST. Then the triangle antecedent to RST is the straight line R_1R, which may be considered as a triangle R_1RR. The side RR of this triangle is infinitely small and its direction is the tangent at A.

(10) Each triangle of the series considered in (7) has its angles equal to half the sum of the angles taken two and two of the preceding triangle. Consider a series of triangles such that each has its sides equal to half the sum of the sides taken two and two of the preceding triangle.

Starting with triangle ABC whose sides are a, b, c, the triangle $A_1B_1C_1$ is to be constructed whose sides are $\frac{1}{2}(b+c)$, $\frac{1}{2}(c+a)$, $\frac{1}{2}(a+b)$.

This second triangle is always possible even when a, b, c are taken at random, provided they be positive. For

$$\frac{c+a}{2} + \frac{a+b}{2} > \frac{b+c}{2}$$

$$\frac{a+b}{2} + \frac{b+c}{2} > \frac{c+a}{2}$$

$$\frac{b+c}{2} + \frac{c+a}{2} > \frac{a+b}{2}$$

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FIGURE 1.

Bisect the sides of ABC at A′, B′, C′. The angular contours A′CB′, B′AC′, C′BA′ straightened out will be the sides of the second triangle $A_1B_1C_1$.

Suppose triangle ABC to be formed by an endless thread which marks out the perimeter. Take the mid points of BC, CA, AB, and stretch the thread between these points, and the second triangle is obtained.

The same process may be repeated on triangle $A_1B_1C_1$ and so on indefinitely. The limiting triangle which is thus obtained may be proved to be the equilateral triangle whose side is $\frac{1}{3}(a+b+c)$.

Can this process be extended backwards indefinitely far? To answer the question a solution must be found for the problem:

Given a triangle whose sides are a, b, c, construct the triangle whose sides are

$$-a+b+c,\ a-b+c,\ a+b-c.$$

FIGURE 24.

In triangle ABC inscribe the circle DEF;

then
$$AE = AF = \frac{-a+b+c}{2}$$

$$BF = BD = \frac{a-b+c}{2}$$

$$CD = CE = \frac{a+b-c}{2}.$$

Hence the triangle whose sides are equal to

$$AE+AF,\ BF+BD,\ CD+CE$$

will be the triangle sought.

Take, as before, the endless thread which marks out the perimeter of ABC at the points D, E, F and stretch it between these points.

Now this triangle is not always possible. For, in order that it may be possible, there must exist the inequality

$$a-b+c+a+b-c > -a+b+c, \text{ or } 3a > b+c.$$

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Similarly $3b > c + a$, and $3c > a + b$.

By the addition of a, b, c these three inequalities may be transformed into $2a > s$, $2b > s$, $2c > s$.

But in every triangle the semiperimeter is greater than any one side. Hence the necessary and sufficient condition that the triangle antecedent to ABC may be possible is that each side of ABC must be greater than a quarter and less than a half of the perimeter.

The whole of (10) and a small part of (9) have been taken from a paper by Mr Édouard Collignon read at the Oran meeting (1888) of the *Association Française pour l'avancement des sciences*. See the Report of this meeting, Second Part, pp. 4-24. Mr Collignon's paper begins with a discussion of certain numerical series, and the results obtained are applied to the triangle, the quadrilateral, and to polygons of any number of sides.

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§ 4. Excentres.

The internal bisector of any angle of a triangle and the external bisectors of the two other angles are concurrent.

The following demonstration is different from the usual one.

Figure 27.

Let AL be the internal bisector of $\angle A$, and let the external bisector of $\angle B$ cut it at I_1.

Then $$AI_1 : LI_1 = BA : BL$$
$$= CA : CL;$$

therefore the external bisector of $\angle C$ passes through I_1.

Hence also the internal bisector of $\angle B$ and the external bisectors of $\angle C$ and $\angle A$ are concurrent at I_2; the internal bisector of $\angle C$ and the external bisectors of $\angle A$ and $\angle B$ are concurrent at I_3.

Figure 28.

The points of concurrency, which will be denoted by I_1, I_2, I_3, are the centres of the circles escribed * to ABC.

These circles are often called the *excircles*,† and the centres of them the excentres.†

The radii of the excircles are denoted sometimes by r_1, r_2, r_3, sometimes by r_a, r_b, r_c.

(1) The points A, I, I_1; B, I, I_2; C, I, I_3 are collinear.
So are I_2, A, I_3; I_3, B, I_1; I_1, C, I_2.

These results expressed in words are:

The six bisectors of the interior and the exterior angles of a triangle meet three and three in four points which are the centres of the four circles touching the sides of the triangle. Or

The six straight lines joining two and two the centres of the four circles which touch the sides of a triangle pass each through one of the vertices of the triangle.

---

* This expression in its French form (*exinscrit*) was first used by Simon Lhuilier. See his *Élémens d'Analyse*, p. 198 (1809). If the term *escribed* was not introduced by T. S. Davies, currency at least was given to it by him. See *Ladies' Diary* for 1835, p. 50.

† See the note on p. 32.

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(2) The points I, I_1, I_2, I_3
are the respective orthocentres of the triangles

$$I_1I_2I_3,\ II_3I_2,\ I_3II_1,\ I_2I_1I.$$

Attention should be called to the order of the subscripts in the naming of the triangles. See § 5, (2).

(3) When the diagram of a triangle with its incentres and excentres has to be drawn, instead of beginning with the triangle ABC and determining I, I_1, I_2, I_3 by the bisection of certain angles, it is easier to begin with the triangle $I_1I_2I_3$. The feet of the perpendiculars of $I_1I_2I_3$ will then be A, B, C, and I will be the point of intersection of the perpendiculars. The only instrument therefore which is necessary to determine these points is a draughtsman's square.

A circle may be escribed to a given triangle by a method exactly analogous to that on p. 39 for inscribing a circle.

FIGURE 24.

The only difference in the construction is that CQ is cut off equal to CB not in the same direction as CP, but in the opposite direction.

(4) The area of a triangle is equal to the rectangle under the excess of the semiperimeter above any side and the radius of the excircle which touches the other two sides produced.

This is expressed, $\triangle = s_1r_1 = s_2r_2 = s_3r_3$
where $s_1 = \frac{1}{2}(-a+b+c)$, $s_2 = \frac{1}{2}(a-b+c)$, $s_3 = \frac{1}{2}(a+b-c)$.

If $\frac{1}{2}(a+b+c) = s$,
then $\frac{1}{2}(-a+b+c) = s-a$, $\frac{1}{2}(a-b+c) = s-b$, $\frac{1}{2}(a+b-c) = s-c$.

The expressions $s-a$, $s-b$, $s-c$ will be denoted by s_1, s_2, s_3, a notation introduced by Thomas Weddle, who in a letter to T. S. Davies, dated March 31st, 1842, and printed in the *Lady's and Gentleman's Diary* for 1843, p. 78, remarks that s, s_1, s_2, s_3 are the lengths of the segments of the sides made by the four circles of contact, and that the change from $s-a$ to s_1, etc., will be justified by observing how much more symmetrical many theorems appear under the new notation than the old.

(5) The distances from the vertices and from each other of some of the points of inscribed and escribed contact are given in the following expressions*:

* Compare the subscripts in the values of s, s_1, s_2, s_3 with the subscripts in § 4 (2).

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FIGURE 28.

$$s = AE_1 = AF_1 = B\,F_2 = BD_2 = C\,D_3 = CE_2$$
$$s_1 = AE = AF = B\,F_3 = BD_3 = C\,D_2 = CE_3$$
$$s_2 = AE_3 = AF_3 = B\,F = BD = C\,D_1 = CE_1$$
$$s_3 = AE_2 = AF_2 = B\,F_1 = BD_1 = C\,D = CE$$

$$a = EE_1 = E_2E_3 = FF_1 = F_3F_2$$
$$b = FF_2 = F_3F_1 = DD_2 = D_1D_3$$
$$c = DD_3 = D_1D_2 = EE_3 = E_2E_1$$

$$b + c = D_2D_3 \quad b \sim c = DD_1$$
$$c + a = E_3E_1 \quad c \sim a = EE_2$$
$$a + b = F_1F_2 \quad a \sim b = FF_3$$

(6) $$a + b + c = 2s = s + s_1 + s_2 + s_3$$
$$= AE + AE_1 + AE_2 + AE_3 = \text{etc.}$$

(7) Because $BD = s_2 = CD_1$

therefore D and D_1 are equidistant from the mid point of BC.

Because $BD_3 = s_1 = CD_2$

therefore D_2 and D_3 are equidistant from the mid point of BC.

Similarly for the E points and the F points.

(8) $$AD^2 + AD_1^2 + AD_2^2 + AD_3^2$$
$$+ BE^2 + BE_1^2 + BE_2^2 + BE_3^2$$
$$+ CF^2 + CF_1^2 + CF_2^2 + CF_3^2$$
$$= 5(a^2 + b^2 + c^2).$$

FIGURE 28.

Let A′ be the mid point of BC;

then A′ is the mid point of DD_1 and of D_2D_3.

Now since $D_2D_3 = b + c \qquad DD_1 = b \sim c,$

therefore $2A'D_2 = b + c \qquad 2A'D = b \sim c.$

But $$AD^2 + AD_1^2 = 2A'D^2 + 2A'A^2$$
$$AD_2^2 + AD_3^2 = 2A'D_2^2 + 2A'A^2;$$

therefore $$\Sigma(AD^2) = 2A'D^2 + 2A'D_2^2 + 4A'A^2.$$

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Again $2AB^2 + 2AC^2 = 4A'B + 4A'A^2$

$= BC^2 + 4A'A^2$;

therefore $2c^2 + 2b^2 - a^2 = 4A'A^2$.

Hence $\Sigma(AD^2) = \frac{1}{2}\{(b \sim c)^2 + (b+c)^2\} + 2c^2 + 2b^2 - a^2$

$= 3(b^2 + c^2) - a^2$.

Similarly $\Sigma(BE^2) = 3(c^2 + a^2) - b^2$

and $\Sigma(CF^2) = 3(a^2 + b^2) - c^2$.

Consequently the sum of the squares on the twelve specified lines

$= 5(a^2 + b^2 + c^2)$.

(8) is stated by W. H. Levy of Shalbourne in the *Lady's and Gentleman's Diary* for 1852, p. 71, and proved in 1853, pp. 52-3.

(9) The angles of triangle DEF expressed in terms of A, B, C are

$$\angle D = \tfrac{1}{2}(B+C) = 90° - \tfrac{1}{2}A$$
$$\angle E = \tfrac{1}{2}(C+A) = 90° - \tfrac{1}{2}B$$
$$\angle F = \tfrac{1}{2}(A+B) = 90° - \tfrac{1}{2}C.$$

Hence whatever be the size of the angles A, B, C the triangle DEF is always acute-angled.*

(10) *If ABC be a triangle, DEF the triangle formed by joining the inscribed points of contact of ABC; $D_1E_1F_1$ the triangle formed by joining the inscribed points of contact of DEF; $D_2E_2F_2$ the triangle formed by joining the inscribed points of contact of $D_1E_1F_1$; and this process of construction be continued, the successive triangles will approximate to an equilateral triangle.*†

Suppose $\angle A$ greater than $\angle B$, and $\angle B$ greater than $\angle C$.

Then $D = \frac{1}{2}(B+C)$, $E = \frac{1}{2}(C+A)$, $F = \frac{1}{2}(A+B)$;

therefore $E - D = \frac{1}{2}(A-B)$, $F - E = \frac{1}{2}(B-C)$, $F - D = \frac{1}{2}(A-C)$.

Now $D_1 = \frac{1}{2}(E+F)$, $E_1 = \frac{1}{2}(F+D)$, $F_1 = \frac{1}{2}(D+E)$;

therefore $D_1 - E_1 = \frac{1}{2}(E-D)$, $E_1 - F_1 = \frac{1}{2}(F-E)$, $D_1 - F_1 = \frac{1}{2}(F-D)$,

$= \frac{1}{4}(A-B)$, $= \frac{1}{4}(B-C)$, $= \frac{1}{4}(A-C)$;

and so on.

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* Feuerbach, *Eigenschaften...des...Dreiecks*, § 66 (1822).

† Todhunter's *Plane Trigonometry*, Chapter XVI. Example 16 (1859).

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Therefore the differences between the angles of the successive triangles become always a smaller and smaller fraction of the differences between the angles of the fundamental triangle; and hence the successive triangles approximate to an equilateral triangle.

The properties (11)–(14), (16)–(18) have reference to Figure 28.

(11) EF, E_1F_1 are parallel to I_2I_3
and EF_1, E_1F intersect on AI; and so on.

(12) E_2F_2, E_3F_3 are parallel to AI
and E_2F_3, E_3F_2 intersect on I_2I_3; and so on.

(13) The angles of triangles $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$ expressed in terms of A, B, C are

$$\begin{aligned}
&\angle D_1 = 90° + \tfrac{1}{2}A, \quad \angle E_1 = \tfrac{1}{2}B, \quad \angle F_1 = \tfrac{1}{2}C\\
&\angle D_2 = \tfrac{1}{2}A, \quad \angle E_2 = 90° + \tfrac{1}{2}B, \quad \angle F_2 = \tfrac{1}{2}C\\
&\angle D_3 = \tfrac{1}{2}A, \quad \angle E_3 = \tfrac{1}{2}B, \quad \angle F_3 = 90° + \tfrac{1}{2}C.
\end{aligned}$$

Hence whatever be the size of the angles A, B, C these three triangles are always obtuse-angled.*

(14) The angles of triangle $I_1I_2I_3$ expressed in terms of A, B, C are

$$\begin{aligned}
\angle I_1 &= \tfrac{1}{2}(B + C) = 90° - \tfrac{1}{2}A\\
\angle I_2 &= \tfrac{1}{2}(C + A) = 90° - \tfrac{1}{2}B\\
\angle I_3 &= \tfrac{1}{2}(A + B) = 90° - \tfrac{1}{2}C
\end{aligned}$$

Hence whatever be the size of the angles A, B, C the triangle $I_1I_2I_3$ is always acute-angled.

(15) If ABC be a triangle, $A_1B_1C_1$ the triangle formed by joining the excentres of ABC; $A_2B_2C_2$ the triangle formed by joining the excentres of $A_1B_1C_1$; and this process of construction be continued, the successive triangles will approximate to an equilateral triangle.†

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* Feuerbach, *Eigenschaften...des...Dreiecks*, § 66 (1822).

† Mr R. Tucker in *Mathematical Questions from the Educational Times*, XV. 103-4 (1871).

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(16) Triangles DEF, $I_1I_2I_3$ are similar and similarly situated,* and their homothetic centre is the point of concurrency of the triad

$$I_1D,\ I_2E,\ I_3F.$$

(17) Triangles $D_1E_1F_1$, II_3I_2 are similar and similarly situated; so are $D_2E_2F_2$, I_3II_1; and $D_3E_3F_3$, I_2I_1I; and their homothetic centres are the points of concurrency of the triads

$$ID_1,\ I_3E_1,\ I_2F_1;\ \text{and so on.}$$

(18) *The quadrilaterals AFIE, BDIF, CEID are such that circles may be inscribed in them.*

For a circle may be inscribed in a quadrilateral when the sum of one pair of opposite sides is equal to the sum of the other pair.

Now $$AF = AE \text{ and } IE = IF.$$

(19) *If the radii of the circles inscribed in the quadrilaterals* † *AFIE, BDIF, CEID* be denoted by ρ_1, ρ_2, ρ_3,

$$\left(\frac{1}{\rho_2}-\frac{1}{r}\right)\left(\frac{1}{\rho_3}-\frac{1}{r}\right)+\left(\frac{1}{\rho_3}-\frac{1}{r}\right)\left(\frac{1}{\rho_1}-\frac{1}{r}\right)+\left(\frac{1}{\rho_1}-\frac{1}{r}\right)\left(\frac{1}{\rho_2}-\frac{1}{r}\right)=\frac{1}{r^2}.$$

FIGURE 29.

Bisect $\angle$ AFI by FM meeting AI at M; and draw MN perpendicular to AF.

Then M is the centre of the circle inscribed in AFIE, MN is the radius of it, and MN = FN.

From the similar triangles AFI, ANM

$$AF : IF = AN : MN$$

that is, $$s_1 : r = s_1 - \rho_1 : \rho_1 \quad ;$$

therefore $$s_1 : r = \rho_1 : r - \rho_1 \ ;$$

therefore $$s_1 = \frac{r\rho_1}{r-\rho_1} \ ;$$

therefore $$\frac{1}{s_1} = \frac{1}{\rho_1} - \frac{1}{r}.$$

Similarly $$\frac{1}{s_2} = \frac{1}{\rho_2} - \frac{1}{r}, \quad \frac{1}{s_3} = \frac{1}{\rho_3} - \frac{1}{r}.$$

Hence given expression $$= \frac{1}{s_2s_3} + \frac{1}{s_3s_1} + \frac{1}{s_1s_2} = \frac{1}{r^2}.$$

* Feuerbach, *Eigenschaften ... des ... Dreiecks*, § 61 (1822).

† *The Museum*, III. 269-70 and 342 (1866).

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(20) $$\frac{\rho_1}{r-\rho_1}+\frac{\rho_2}{r-\rho_2}+\frac{\rho_3}{r-\rho_3}=\frac{\rho_1}{r-\rho_1}\cdot\frac{\rho_2}{r-\rho_2}\cdot\frac{\rho_3}{r-\rho_3}.$$

For $$\frac{r\rho_1}{r-\rho_1}=s_1;$$

therefore $$\frac{\rho_1}{r-\rho_1}=\frac{s_1}{r}.$$

Similarly $$\frac{\rho_2}{r-\rho_2}=\frac{s_2}{r},\quad \frac{\rho_3}{r-\rho_3}=\frac{s_3}{r};$$

therefore $$\frac{\rho_1}{r-\rho_1}+\frac{\rho_2}{r-\rho_2}+\frac{\rho_3}{r-\rho_3}=\frac{s_1+s_2+s_3}{r}=\frac{s}{r}$$

and $$\frac{\rho_1}{r-\rho_1}\cdot\frac{\rho_2}{r-\rho_2},\frac{\rho_3}{r-\rho_3}=\frac{s_1s_2s_3}{r^3}=\frac{s}{r}.$$

(21) *The quadrilaterals* $AF_1I_1E_1$, $BD_1I_1F_1$, $CE_1I_1D_1$ *are such that circles may be inscribed in them.*

FIGURE 28.

For $$AF_1=AE_1 \text{ and } I_1E_1=I_1F_1.$$

Similarly, circles may be inscribed in the quadrilaterals

$$AF_2I_2E_2,\ BD_2I_2F_2,\ CE_2I_2D_2$$
$$AF_3I_3E_3,\ BD_3I_3F_3,\ CE_3I_3D_3.$$

(22) If the radii of the circles inscribed in the first three of these quadrilaterals be denoted by ρ_1', ρ_2', ρ_3' then

$$\left(\frac{1}{\rho_2'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_3'}-\frac{1}{r_1}\right)-\left(\frac{1}{\rho_3'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_1'}-\frac{1}{r_1}\right)-\left(\frac{1}{\rho_1'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_2'}-\frac{1}{r_1}\right)=\frac{1}{r_1^2};$$

and similarly for the others.

(23) $$\frac{\rho_1'}{r_1-\rho_1'}-\frac{\rho_2'}{r_1-\rho_2'}-\frac{\rho_3'}{r_1-\rho_3'}=\frac{\rho_1'}{r_1-\rho_1'}\cdot\frac{\rho_2'}{r_1-\rho_2'}\cdot\frac{\rho_3'}{r_1-\rho_3'}$$

For $$s=\frac{r_1\rho_1'}{r_1-\rho_1'},\quad s_3=\frac{r_1\rho_2'}{r_1-\rho_2'},\quad s_2=\frac{r_1\rho_3'}{r_1-\rho_3'}.$$

(24) The following relation* exists between the radii of the circles inscribed in the quadrilaterals

$$AF_1I_1E_1,\ BD_2I_2F_2,\ CE_3I_3D_3.$$

* Mr R. E. Anderson in *Proceedings of the Edinburgh Mathematical Society*, X. 79 (1892).

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If these radii be denoted by v_1, v_2, v_3

$$\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}=\frac{3}{s}+\frac{1}{r}.$$

(25) *If the exterior angles of ABC be bisected by straight lines which meet the circumcircle at U', V', W', the sides of U'V'W' are also perpendicular to AI, BI, CI, and U'V'W' is congruent and symmetrically situated to UVW with respect to the circumcentre of ABC.*

FIGURE 25.

For $\angle UAU'$ is right;
therefore UU' is a diameter of the circumcircle,
and U' is symmetrical to U with respect to O.

(26) If from the six points U, V, W, U', V', W' all the UVW triangles be formed, it will be found that there are four pairs

$$\begin{matrix} UVW, & UV'W', & U'VW', & U'V'W \\ U'V'W', & U'VW, & UV'W, & UVW'. \end{matrix}$$

These pairs of triangles are congruent and symmetrically situated with respect to O, and their sides are either perpendicular or parallel to AI, BI, CI.

(27) The angles of these triangles can be expressed in terms of A, B, C.

Take, for example, triangle U'VW from the second pair.

$$WU'V = 180° - WUV$$
$$= 180° - \tfrac{1}{2}(B+C) = 90° + \tfrac{1}{2}A.$$

Since AU', VW are both perpendicular to AI,

therefore arc U'W = arc AV
therefore $\angle U'VW = \angle ABV = \tfrac{1}{2}B$,
and $\angle U'WV = \angle ACW = \tfrac{1}{2}C$.

The angles of the third and fourth pairs of triangles are respectively equal to

$$\begin{matrix} \tfrac{1}{2}A, & 90°+\tfrac{1}{2}B, & \tfrac{1}{2}C \\ \tfrac{1}{2}A, & \tfrac{1}{2}B, & 90°+\tfrac{1}{2}C. \end{matrix}$$

Hence whatever be the size of the angles A, B, C the second, third, and fourth pairs of triangles are always obtuse-angled.

Compare § 4, (13).

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(28) The orthocentres of the first quartet of triangles are

$$I,\ I_1,\ I_2,\ I_3$$

and the orthocentres of the second quartet are the points symmetrical to I, I_1, I_2, I_3 with respect to O. What these points are will be seen later on.

(29)
$$\begin{aligned} U\,V'W' : ABC &= R : 2r_1 \\ U'V\,W' : ABC &= R : 2r_2 \\ U'V'W\ : ABC &= R : 2r_3. \end{aligned}$$

(30) If I be the incentre of ABC and about the triangles IBC, ICA, IAB circles be circumscribed, these circles will pass respectively through I_1, I_2, I_3, and their centres will be U, V, W, the points where AI, BI, CI meet the circumcircle.

FIGURE 28.

(31) If circles be circumscribed about the triangles I_1BC, I_1CA, I_1AB, these circles will pass respectively through I, I_3, I_2.

Similarly for the circumcircles of I_2BC, etc.

(32) From (30) and (31) there would seem to be twelve circles. They reduce to six, and their diameters are the six lines

$$II_1,\ II_2,\ II_3,\ I_2I_3,\ I_3I_1,\ I_1I_2.$$

(33) *In a triangle ABC, if on AB and AC as diameters circles be described and a diameter of the first circle be drawn parallel to AC, and a diameter of the second parallel to AB, one pair of extremities of these diameters will lie on the internal bisector of angle A and the other pair on the external bisector.**

FIGURE 30.

Let M and N be the centres of the circles described on AB and AC, and let EF be the diameter parallel to AC, and GH the diameter parallel to AB. Join AF.

Then $\angle MAF = \angle MFA = \angle NAF$;

therefore AF is the internal bisector of $\angle A$.

Similarly AH is the internal bisector of $\angle A$.

Join AE, and produce CA to C'.

* Mr Brocot in the *Journal de Mathématiques Élémentaires*, I. 383 (1877), II. 128 (1878).

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Then $\angle MAE = \angle MEA = \angle C'AE$;
therefore AE is the external bisector of $\angle A$.
Similarly AG is the external bisector of $\angle A$.

(34) *If M, N be the feet of the bisectors of angles B and C of triangle ABC, the distance of any point P in MN from BC is equal to the sum* of its distances from CA and AB.*

FIGURE 31.

Draw MK, MK′ perpendicular to BC, AB
NL, NL′ „ „ BC, CA;
join ML meeting PR in H.

Then $PS : NL' = PM : NM$
$= PH : NL.$
But $NL' = NL$;
therefore $PS = PH$.
Again $PT : MK' = PN : MN$
$= RL : KL$
$= HR : MK.$
But $MK' = MK$;
therefore $PT = HR$;
therefore $PR = PS + PT.$

If P be situated on MN produced either way, then PR is equal to the difference between PS and PT.

The theorem may be extended to the bisectors of the exterior angles of ABC, and thus enunciated:

If M, N be the feet of the internal or external bisectors of the angles B and C, the distance of any point P in MN from BC is equal to the algebraic sum of its distances from CA and AB.

(35) If through I, I_1, I_2, I_3 parallels be drawn to BC meeting AC, AB in P, Q; P_1, Q_1; P_2, Q_2; P_3, Q_3
then $PQ = BQ + CP$, $P_1Q_1 = BQ_1 + CP_1$
$P_2Q_2 = BQ_2 - CP_2$, $P_3Q_3 = BQ_3 - CP_3$.

* Mr E. Cesaro in *Nouvelle Correspondance Mathématique*, V. 224 (1879); proof and extension of the property to the external bisectors by Mr Cauret on pp. 334-5. The proof in the text is taken from Vuibert's *Journal* IX. 72 (1885). Mr Cesaro gives the corresponding property for the tetrahedron.

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(36) *If AI meet the incircle at U, and at U a tangent be drawn meeting BI and CI at P and Q, then**

$$BP = CQ = BI + CI.$$

FIGURE 32.

Let PQ meet CA, AB at S, T.

Then $\angle QTB = \angle ATS$
$= \frac{1}{2}(B + C)$;

and $\angle QIB = \angle IBC + \angle ICB$
$= \frac{1}{2}(B + C)$;

therefore the points B, I, T, Q are concyclic,

and $\angle IQU = \angle IBF$.

Hence the right-angled triangles IUQ, IFB, which have IU equal to IF, are congruent;

therefore $QI = BI$.

Similarly $PI = CI$;

therefore $BP = CQ = BI + CI$.

(37) $BT = QS$ and $CS = PT$.

For $BT = BF + TF = QU + TU = QU + SU$.

(38) If AI_1 meet the first excircle at U_1, and at U_1 a tangent be drawn meeting BI_1 and CI_1 at P_1 and Q_1, then

$$BP_1 = C\,Q_1 = BI_1 + CI_1.$$

(39) $PQ = P_1Q_1 = BC$.

(40) $BT_1 = Q_1S_1$ and $CS_1 = P_1T_1$.

(41) *If ID, the radius of the incircle, meet EF at P, then* † *P lies on the median through A.*

FIGURE 33.

Through P draw B_1C_1 parallel to BC, and join IB_1, IC_1.

Then $\angle IPB_1$ and $\angle IFB_1$ are right;

therefore the points I, P, F, B_1 are concyclic;

therefore $\angle IB_1P = \angle IF\,P$.

---

* This and (37) are given by William Wallace in Leybourn's *Mathematical Repository*, old series, II. 187 (1801).

† John Johnson, of Birmingham, in Leybourn's *Mathematical Repository*, old series, II. 376 (1801).

Mr E. M. Langley in the *Sixteenth Report of the Association for the Improvement of Geometrical Teaching*, pp. 35-6 (1890), gives another demonstration by means of Brianchon's theorem:

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Similarly $\angle IC_1P = \angle IEP$;

therefore $\angle IB_1P = \angle IC_1P$;

therefore $B_1P = C_1P$.

Hence also if I_1D_1, the radius of the first excircle, meet E_1F_1 at P_1, then P_1 lies on the median through A.

(42) *If from any point P situated on the interior or exterior bisector of the angle A of triangle ABC perpendiculars PD, PE, PF be drawn to BC, CA, AB, the point Q where PD intersects EF will lie on the median* from A.*

FIGURE 34.

Triangles FPQ, ABL are similar, since their sides are mutually perpendicular;

therefore $FQ : FP = AL : AB$.

Similarly $EQ : EP = AL : AC$;

therefore $AB \cdot FQ = AC \cdot EQ$.

Now FQ and EQ are proportional to the distances of Q from AB and AC;

therefore $ABQ = ACQ$.

But these triangles have the same base AQ;

therefore their corresponding altitudes are equal;

and hence it is easily deduced that AQ passes through the mid point of BC.

The following is another demonstration:

When the point P moves on the bisector, the point Q describes a straight line passing through A.

Place the point P at the intersection of the bisector with the circumcircle of ABC;

then the projections of P on the sides of ABC are collinear, by Wallace's theorem;

and one of these projections is the mid point of BC.

* Mr E. Cesaro in *Mathesis* I. 79 (1881). The two demonstrations are from the same volume, pp. 117-8.

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(43) *The problem of finding the incentre or the excentres of a triangle is a particular case of the problem to find a point such that straight lines drawn from it to the sides shall make equal angles with the sides and shall be to each other in given ratios.**

FIGURE 35.

Let the straight lines drawn from the point to BC, CA, AB be in the ratios $d : e : f$.

Make a parallelogram having B for one of its angles, and having the sides along BA, BC in the ratio $d : f$;
let BM be one of its diagonals.

Make a parallelogram having C for one of its angles, and having the sides along CA, CB in the ratio $d : e$;
let CN be one of its diagonals.
Then BM, CN will meet at I the required point.

From I draw ID, IE, IF making with BC, CA, AB angles equal to the given angle.

The proof will offer no great difficulty if from M perpendiculars be drawn to BA, BC, from N to CA, CB, and from I to the three sides.

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* Mauduit's *Leçons de Géométrie*, pp. 239-242 (1790).

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§ 5. Orthocentre.

*The perpendiculars to the sides of a triangle from the opposite vertices are concurrent.**

One of the earliest demonstrations occurs in Pierre Herigone's *Cursus Mathematicus*, I. 318 (1634). Three cases are considered, when the triangle is right-angled, acute-angled, obtuse-angled.

From the various proofs that have been published, the following are selected.

First Demonstration.†

Figure 36.

Let AX, BY which are perpendicular to BC, CA meet at H, and let CH be joined and produced to meet AB at Z.

Join XY.

Because $\angle AXC$ and $\angle BYC$ are right,

therefore C, X, H, Y are concyclic, as well as A, Y, X, B;

therefore $\angle ACZ = \angle AXY,$

$= \angle ABY.$

Now $\angle ZAY$ is common to triangles ACZ, ABY;

therefore $\angle AZC = \angle AYB,$

= a right angle.

Second Demonstration.‡

Figure 37.

Let AX, BY, CZ be the three perpendiculars from A, B, C on BC, CA, AB.

Through A, B, C draw B_1C_1, C_1A_1, A_1B_1 respectively parallel to BC, CA, AB.

---

* This theorem occurs without proof in the fifth of the *Lemmas* ascribed to Archimedes, and also in Pappus's *Mathematical Collection*, VII. 62. In Commandino's editions of Pappus, which were published after his death, the proof supplied is erroneous. The mistake has been noticed by several mathematical writers.

† Robert Simson's *Opera Quaedam Reliqua*, p. 171 (1776).

‡ This mode of proof is given by F. J. Servois in his *Solutions peu connues de différens problèmes de Géométrie-pratique*, p. 15 (1804). It was also given by Gauss, and will be found in Schumacher's translation into German of Carnot's *Géométrie de Position*, II. 363 (1810).

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Then $ABCB_1$, $ACBC_1$ are parallelograms,
and A is the mid point of B_1C_1.
Hence also B and C are the mid points of C_1A_1 and A_1B_1.
But AX, BY, CZ are respectively perpendicular to BC, CA, AB;
therefore they must be respectively perpendicular to B_1C_1, C_1A_1, A_1B_1.
If therefore it be assumed as true that the perpendiculars to the sides of a triangle from the mid points of the sides are concurrent,

AX, BY, CZ are concurrent.

Third Demonstration.*

Figure 38.

Let AX, BY, CZ be the three perpendiculars from A, B, C on BC, CA, AB.

Join YZ, ZX, XY.

Since the points A, Z, X, C are concyclic,

therefore $\angle BXZ = \angle BAC$.

Since the points A, Y, X, B are concyclic,

therefore $\angle CXY = \angle BAC$;

therefore $\angle BXZ = \angle CXY$.

Now $\angle BXA = \angle CXA$;

therefore AX bisects $\angle ZXY$.

Hence BY „ $\angle XYZ$,

and CZ „ $\angle YZX$.

If therefore it be assumed as true that the internal angular bisectors of a triangle are concurrent

AX, BY, CZ are concurrent.

Fourth Demonstration.†

"If three straight lines drawn through the vertices of a triangle are concurrent, their isogonals with respect to the angles of the triangle are also concurrent."

This theorem, which is due to Steiner,‡ taken along with the property, which is established in the proof of Brahmegupta's theorem, namely,

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* Mr Bernh. Möllmann in Grunert's *Archiv*, XVII., 376 (1851).

† Dr James Booth's *New Geometrical Methods*, II. 260-1 (1877).

‡ Gergonne's *Annales*, XIX. 37-64 (1828), or Steiner's *Gesammelte Werke*, I. 193 (1881).

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"The perpendicular from any vertex of a triangle to the opposite side and the diameter of the circumcircle drawn from that vertex are isogonal with respect to the vertical angle" furnishes a ready proof. For the diameters of the circumcircle are concurrent.

The point H, where AX, BY, CZ are concurrent, is now generally called the orthocentre* of ABC; and the triangle XYZ is called sometimes the orthic,† sometimes the orthocentric,‡ and sometimes the pedal, triangle.

It may be noted that H is the initial letter in English, French, and German of the names for AX, BY, CZ (*Heights*, *Hauteurs*, *Höhen*).

(1) If in Fig. 37 ABC be considered the fundamental triangle, $A_1B_1C_1$ is anticomplementary to it, and hence the orthocentre of any triangle is the circumcentre of the anticomplementary triangle.

If however $A_1B_1C_1$ be considered the fundamental triangle, ABC is complementary to it, and hence the circumcentre of any triangle is the orthocentre of the complementary triangle.

(2) The four points A, B, C, H, taken three by three form four triangles ABC, HCB, CHA, BAH; of these four triangles the fourth points H, A, B, C are the respective orthocentres, and in all the four cases the orthic triangle is XYZ. "The figure is therefore a system of four points joined two and two by straight lines such that each of them passing through two of these points cuts perpendicularly that which passes through the two others."§

In naming the four triangles the order of the letters is such that X is the foot of the perpendicular from the vertex first named, Y the foot of that from the second named vertex, and Z the foot of that from the third. This is a matter of much more importance than appears at first sight.

It may be convenient to call a set of four points such as A, B, C, H an *orthic tetrastigm*.

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* This useful expression was suggested by Dr Ferrers and Dr W. H. Besant in 1866-7. It is introduced in Dr Besant's *Conic Sections*, § 138 (1869).

† Mr Emile Vigarié in *Mathesis*, VII. 61 (1887).

‡ Dr James Booth in his *New Geometrical Methods*, II. 261 (1877).

§ Carnot, *Corrélation des Figures de Géométrie*, § 143 (1801).

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(3) The angles of the triangles HCB, CHA, BAH expressed in terms of A, B, C are

$$\angle BHC = 180° - A, \quad \angle HCB = 90° - B, \quad \angle CBH = 90° - C$$
$$\angle ACH = 90° - A, \quad \angle CHA = 180° - B, \quad \angle HAC = 90° - C$$
$$\angle HBA = 90° - A, \quad \angle BAH = 90° - B, \quad \angle AHB = 180° - C.$$

(4) *The fundamental triangle is inversely similar to the triangles "cut off" from it by the sides of the orthic triangle.*

FIGURE 38.

If ABC be the fundamental triangle, H is its orthocentre, XYZ its orthic triangle, and the triangles cut off from ABC and similar to it are AYZ, XBZ, XYC.

If HCB be taken as the fundamental triangle, A is its orthocentre, XYZ its orthic triangle, and the triangles "cut off" from HCB and similar to it are HYZ, XCZ, XYB.

Similarly for CHA and triangles CYZ, XHZ, XYA

and for BAH „ „ BYZ, XAZ, XYH.

(5) ABC is the orthic triangle not only of $I_1I_2I_3$, but also of II_3I_2, I_3II_1, I_2I_1I.

FIGURE 28.

Hence the sides of ABC "cut off" from these four triangles four triads of triangles which are respectively similar to them. They are

To $I_1I_2I_3$; I_1BC, AI_2C, ABI_3
„ II_3I_2 ; IBC, AI_3C, ABI_2
„ I_3II_1 ; I_3BC, AIC, ABI_1
„ I_2I_1I ; I_2BC, AI_1C, ABI.

(6) The following triads of lines form by their intersections four triangles which are similar and oppositely situated to the four triangles of the orthic tetrastigm $II_1I_2I_3$.

FIGURE 28.

Lines.	Triangles.
E_1F_1, F_2D_2, D_3E_3	$I_1I_2I_3$
EF, F_3D_3, D_2E_2	II_3I_2
E_3F_3, FD, D_1E_1	I_3II_1
E_2F_2, F_1D_1, DE	I_2I_1I

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Compare the subscripts in the naming of the lines with the subscripts in the naming of the triangles.

(7) *The sides of the orthic triangle are respectively antiparallel* * *to those of the fundamental triangle with respect to the angles of the fundamental triangle.*

FIGURE 38.

If ABC be taken as the fundamental triangle,

YZ is antiparallel to BC with respect to $\angle$ CAB,
ZX „ „ „ CA „ „ „ $\angle$ ABC
XY „ „ „ AB „ „ „ $\angle$ BCA.

If HCB be taken as the fundamental triangle,

YZ is antiparallel to CB with respect to $\angle$ BHC
ZX „ „ „ BH „ „ „ $\angle$ HCB
XY „ „ „ HC „ „ „ $\angle$ CBH.

Similarly for the triangles CHA, BAH.

(8) The angles of triangle XYZ expressed in terms of A, B, C are:

$$\angle X = 180° - 2A = -A + B + C$$
$$\angle Y = 180° - 2B = A - B + C$$
$$\angle Z = 180° - 2C = A + B - C.$$

(9) If ABC, XYZ, $X_1Y_1Z_1$, $X_2Y_2Z_2$ ..... be a series of triangles such that each is the orthic triangle of the preceding, the following tabular statements of their angles may be given.*

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* Carnot's *Géométrie de Position*, § 151 (1803). The term antiparallel was first used by Antoine Arnauld. See *Nouveaux Éléments de Géométrie*, par Messrs de Port-Royal, p. 212, or livre onzième (1667). Further information regarding the use of the word will be found in two letters from Mr E. M. Langley to *Nature*, XL., 460-1 (1889), and XLI., 104-5 (1889).

* These are taken from an article by Mr H. Brocard in the *Nouvelle Correspondance Mathématique*, VI. 145-151 (1880).

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Triangles.	Angles.		
A B C	A	B	C
X Y Z	$-A+B+C$	$A-B+C$	$A+B-C$
$X_1Y_1Z_1$	$3A-B-C$	$-A+3B-C$	$-A-B+3C$
$X_2Y_2Z_2$	$-5A+3B+3C$	$3A-5B+3C$	$3A+3B-5C$
$X_3Y_3Z_3$	$11A-5B-5C$	$-5A+11B-5C$	$-5A-5B+11C$
.....	.....	.....	.....

Consider the coefficients (all taken with the positive sign) of the angle A in the first column of angles. They form the series

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7
1	3	5	11	21	43	85	171

where the law of recurrence is

$$u_{n+1} = u_n + 2u_{n-1}$$

with the initial conditions $u_0 = 1$, $u_1 = 3$.

Triangles.	Angles.		
A B C	A	B	C
X Y Z	$\pi - 2A$	$\pi - 2B$	$\pi - 2C$
$X_1Y_1Z_1$	$4A - \pi$	$4B - \pi$	$4C - \pi$
$X_2Y_2Z_2$	$3\pi - 8A$	$3\pi - 8B$	$3\pi - 8C$
$X_3Y_3Z_3$	$16A - 5\pi$	$16B - 5\pi$	$16C - 5\pi$
.....	.....	.....	.....

In these expressions the coefficient of A, B, or C is a power of 2, and the coefficient of π is one term of the series $u_0\ u_1\ u_2\ u_3 \ldots$

The angle $X_{2n} = u_{2n-1}\pi - 2^{2n+1}A$;

" " $X_{2n-1} = 2^{2n}A - u_{2n-2}\pi$.

(10) *The orthocentre and the vertices of the fundamental triangle are the incentre and the excentres of the orthic triangle.**

* Feuerbach, *Eigenschaften...des...Dreiecks*, § 24 (1822).

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FIGURE 38.

In Möllmann's demonstration of the concurrency of the perpendiculars, it was shown that, if ABC be taken as the fundamental triangle, H is the incentre of XYZ.

Now since BC, CA, AB are respectively perpendicular to AX, BY, CZ, therefore BC, CA, AB are the bisectors of the external angles of XYZ;
therefore A, B, C are the excentres of XYZ.

If HCB be taken as the fundamental triangle, its vertices, B, C and its orthocentre A are the excentres of XYZ, and the vertex H is the incentre.

Similarly for the triangles CHA, BAH.

(11) *If from the mid points of YZ, ZX, XY perpendiculars be drawn to BC, CA, AB, these perpendiculars are concurrent.**

If X', Y', Z' be the mid points,
then triangle X'Y'Z' is similar and oppositely situated to XYZ;
therefore the respective perpendiculars are the bisectors of the angles of X'Y'Z', and consequently concurrent at the incentre of X'Y'Z'.

(12) The perpendiculars from X', Y', Z' respectively to

CB, BH, HC HA, AC, CH AH, HB, BA	are concurrent at the	first excentre of X'Y'Z' second „ „ „ third „ „ „

These four points, the incentre and the excentres of triangle X'Y'Z', will be considered again, in connection with the Taylor circles.

(13) *If the perpendiculars of a triangle meet the circumcircle again in R, S, T, then R, S, T are the images of the orthocentre in the sides.*

FIGURE 39.

Let ABC be the triangle, H its orthocentre.

Join BR.

Then $\angle CBY = \angle CAX = \angle CBR$;

therefore the right-angled triangles BXH, BXR are congruent,

and $HX = RX$.

Similarly $HY = SY$ and $HZ = TZ$.

* Édouard Lucas in *Nouvelle Correspondance Mathématique*, II. 95, 218 (1876).

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If HCB be taken as the triangle instead of ABC, then A is its orthocentre, HX, CY, BZ its perpendiculars. Let a circle be circumscribed about HCB, and let the perpendiculars meet it again at R_1, S_1, T_1.

FIGURE 40.

Then it may be shown as before that

$$AX = R_1X, \quad AY = S_1Y, \quad AZ = T_1Z.$$

Similarly for the triangles CHA, BAH.

(14) *The triangles RST, XYZ are similar and similarly situated; H is their homothetic centre, and their ratio of similitude is* 2 : 1.

FIGURE 40.

Since X, Y, Z are the mid points of HR, HS, HT, therefore the sides of XYZ are respectively parallel to those of RST, and equal to the halves of them.

In like manner the triangles $R_1S_1T_1$, XYZ are similar and similarly situated; A is their homothetic centre, and their ratio of similitude is 2 : 1.

(15) H is the incentre of RST,
A „ „ first excentre „ $R_1S_1T_1$.

Similarly for B and C.

(16) *The circumcircle of ABC is equal* to the circumcircles of HCB, CHA, BAH.*

FIGURE 39.

For triangle HCB is congruent to RCB;
and the circumcircle of RCB is the circumcircle of ABC.

(17) *If O_a, O_b, O_c be the centres of the circumcircles of HCB, CHA, BAH, then triangle $O_aO_bO_c$ is congruent, and oppositely situated, to ABC.*

FIGURE 41.

For O_bO_c, O_cO_a, O_aO_b are perpendicular to HA, HB, HC
and BC, CA, AB „ „ „ „ „ „ .

* Carnot's *Corrélation des Figures de Géométrie*, § 146 (1801), or *Géométrie de Position*, § 130 (1803).

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(18) *H is the circumcentre of* $O_aO_bO_c$,
O „ „ *orthocentre* „ „ .

Since the circles O_b, O_c are equal,
therefore O_bO_c bisects, and is bisected by, their common chord HA perpendicularly;
therefore $HO_b = HO_c$.
Similarly $HO_c = HO_a$.

Again, since the circles O, O_a are equal,
therefore OO_a bisects, and is bisected by, their common chord BC perpendicularly;
therefore O_aO is perpendicular to O_bO_c.
Similarly O_bO „ „ „ O_cO_a.

(19) The points O_a, O_b, O_c, O form an orthic tetrastigm, congruent and oppositely situated to the orthic tetrastigm A, B, C, H.

(20) If through A any straight line be drawn meeting the circles O_b, O_c in M, N, then MC, NB will meet on the circumference of O_a.

(21) If any point L be taken on the circumference of O_a, and LC, LB meet the circumferences of O_b, O_c again in M, N, then the points M, A, N are collinear, and triangle LMN is directly similar to ABC.

(22) Of all the triangles such as LMN whose sides pass through A, B, C, and whose vertices are situated on the circles O_a, O_b, O_c, that triangle $A_1B_1C_1$ is a maximum whose sides are perpendicular to AH, BH, CH.

Compare § 2, (15) – (19).

(23) *Triangle* $A_1B_1C_1$ *is the anticomplementary triangle of ABC; it has H for its circumcentre, and its circumcircle touches the circles* O_a, O_b, O_c *at the points* A_1, B_1, C_1.

For A, B, C are the mid points of B_1C_1, C_1A_1, A_1B_1;
and H, O_a, A_1; H, O_b, B_1; H, O_c, C_1 are collinear.

(24) What has been already proved with regard to the triangle ABC, its orthocentre H, its circumcentre O, and the circles O_a, O_b, O_c may be applied, with the necessary modifications, to the triangle HCB, its orthocentre A, its circumcentre O_a, and the circles O, O_c, O_b; and to the triangles CHA, BAH.

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(25) *If RT, RS meet BC at D, D'; SR, ST meet CA at E, E'; TS, TR meet AB at F, F', then*

HDRD', HESE', HFTF' are rhombi,

and *D, H, E'; E, H, F'; F, H, D';*

*D', H, F; E', H, D; F', H, E are collinear.**

FIGURE 42.

Since HR is bisected perpendicularly by DD',
therefore HD = RD and HD' = RD'.
But since XY, XZ make equal angles with BC,
and RS, RT are respectively parallel to XY, XZ;
therefore RD = RD', and HDRD' is a rhombus.

Again since DH is parallel to RD'
and HE' ,, ,, ,, ES;
therefore D, H, E' are collinear.

(26) *If R_1T_1, R_1S_1 meet CB at D, D'; S_1R_1, S_1T_1 meet BH at E', E; T_1S_1, T_1R_1 meet HC at F', F, then*

ADR_1D', AES_1E', AFT_1F' are rhombi,

and D, A, E; E', A, F; F', A, D', etc., are collinear.

FIGURE 40.

Two other triads of rhombi, and of collinear points may be obtained from triangles CHA, BAH.

(27) *If U, V, W be the mid points of AH, BH, CH, then U, V, W are the orthocentres of triangles AC'B', C'BA', B'A'C.*

FIGURE 43.

For the perpendicular from B' to AC' is parallel to CH;
and since B' is the mid point of AC, this perpendicular passes through the mid point of AH, that is U;
and AU is perpendicular to C'B'.

(28) *The points U, V, W, H form an orthic tetrastigm, where H is the orthocentre of UVW.*

* *Nouvelles Annales*, 2nd series, XIX. 176 (1880) and 3rd series, I. 186–9 (1882).

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If the triangle UVW be translated so that U moves along UA and VW remains parallel to BC, it will coincide with triangle AC'B'.

Similarly the triangle UVW may be made to coincide with C'BA' and B'A'C.

(29) FIGURE 43.

$$\left.\begin{matrix} U, B', C' \\ A', V, C' \\ A', B', W \end{matrix}\right\} \text{ are orthocentres of } \left\{\begin{matrix} HWV, CA'W, BVA' \\ CWB', HUW, AB'U \\ BC'V, AUC', HVU. \end{matrix}\right.$$

(30) *The point H may be the orthocentre of an infinite number of triangles inscribed in the circle ABC.*

FIGURE 39.

For, take any point A on the circumference;
and draw the chord AHR.
Bisect HR at X, and through X draw the chord BC perpendicular to AR.

Then ABC is a triangle whose orthocentre is H.

(31) *The point A may be the orthocentre of an infinite number of triangles inscribed in the circle HCB.*

FIGURE 40.

For, take any point H on the circumference;
and draw the secant AHR_1.
Bisect AR_1 at X, and through X draw the chord BC perpendicular to AR_1.

Then HCB is a triangle whose orthocentre is A.

Similarly for B and C.

(32) *The straight lines joining the circumcentre to the vertices of a triangle are perpendicular to the sides of the orthic triangle; and the straight lines joining the orthocentre to the vertices are perpendicular to the sides of the complementary triangle; and conversely.*

FIGURE 44.

Let OA meet YZ at X';
from O draw OB' perpendicular to CA;
from B' draw B'C' parallel to BC.

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Then B′ is the mid point of CA,
and B′C′ is a side of the complementary triangle.

Hence $\angle AOB' = \angle ABC = \angle AYX'$;
therefore $\angle AX'Y = \angle AB'O =$ a right angle;
and HA is perpendicular to B′C′.

This theorem will be found to be a particular case of a more general one regarding isogonal lines.

(33) The straight lines joining the circumcentre to the vertices of a triangle are perpendicular to all straight lines which are anti-parallel to the sides with respect to the opposite angles; and the straight lines joining the orthocentre to the vertices are perpendicular to all straight lines which are parallel to the sides.

(34) *The straight lines AX and AO are isogonals* with respect to angle BAC.*

FIGURE 44.

For $\angle ABX = \angle AOB'$,
$\angle AXB = \angle AB'O$;
therefore $\angle XAB = \angle OAC$.

Similarly BY, BO are isogonals with respect to $\angle B$.
and CZ, CO „ „ „ „ „ $\angle C$.

The theorem may be stated and proved otherwise, thus:

The straight lines joining the incentre with the vertices of a triangle bisect the angles between the radii of the circumcircle drawn to the vertices and the perpendiculars.

FIGURE 45.

Produce AI to meet the circumcircle in U, and join OU.

Because AU bisects $\angle BAC$,
therefore U is the mid point of the arc BUC;
therefore OU is perpendicular to BC;
therefore $\angle XAU = \angle OUA = \angle OAU$.

* This corollary is established in the proof of the theorem known as Brahmegupta's.

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(35) *The straight lines joining the mid points of*

$$AH,\ BC\ ;\quad BH,\ CA\ ;\quad CH,\ AB$$

make with

$$AB\ ,\qquad BC\ ,\qquad CA$$

angles complementary to*

$$C\ ,\qquad A\ ,\qquad B.$$

FIGURE 43.

Let A′, U be the mid points of BC, AH, and O the circumcentre. Join OA.

Then OA′ is equal and parallel to AU ;
therefore OA is parallel to A′U.
Now OA makes with AB an angle equal to CAH,
that is, an angle complementary to C ;
therefore A′U makes with AB an angle complementary to C.

(36) *The same straight lines make with AX, BY, CZ angles equal to* $B \sim C$, $C \sim A$, $A \sim B$.

For
$$\begin{aligned} \angle \mathrm{A'UX} &= \angle \mathrm{OAX} \\ &= \angle \mathrm{BAX} - \angle \mathrm{CAX} \\ &= \mathrm{C} - \mathrm{B}. \end{aligned}$$

(37) *The angle between*†

$$B'Z \text{ and } C'Y = 3A, \qquad C'X \text{ and } A'Z = 3B, \qquad A'Y \text{ and } B'X = 3C.$$

FIGURE 46.

Produce BY to B_1 so that $B_1Y = BY$,
and „ CZ „ C_1 „ „ $C_1Z = CZ$,
and join AB_1, AC_1.

Then $\angle B_1AY$ and $\angle C_1AZ$ are each equal to A ;
therefore $\angle B_1AC_1 = 3A$.
But since C′ and Y are the mid points of BA and BB_1,
therefore C′Y is parallel to AB_1.
Similarly B′Z „ „ „ AC_1;

* Dr C. Taylor in *Mathematical Questions from the Educational Times*, XVIII. 65 (1872).

† This property and the demonstration of it are due to Professor R. E. Allardice.

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therefore the angle between B′Z and C′Y is equal to the angle between AB_1 and AC_1.

(38) *The straight lines drawn from the orthocentre of a triangle through the mid points of the sides and terminated by the circumcircle are bisected by the sides.*

FIGURE 47.

Let ABC be the triangle, H its orthocentre.

Draw CL parallel to HB and terminated by the circumcircle. Join BL.

Because CL is parallel to AB,

therefore $\angle ACL$ is right;

therefore $\angle ABL$ „ „ ;

therefore BL is parallel to HC.

Hence HBLC is a parallelogram, and its diagonals bisect each other;

that is, HL drawn through A′, the mid point of BC, is bisected by BC.

This corollary may be used to prove part of the characteristic property of the nine-point circle.

(39) $A'Y = A'Z,\ B'Z = B'X,\ C'X = C'Y.$

FIGURE 47.

For B, Z, Y, C are situated on the circumference of a circle whose centre is A′.

(40) *If on each side of a triangle as diagonal two parallelograms be constructed, the one having a vertex at the opposite angle of the triangle, the other at the centre of the circumcircle, then the straight lines which join the other vertices of these three pairs of parallelograms will pass through the orthocentre.**

FIGURE 48.

FIRST DEMONSTRATION.

Let H be the orthocentre, O the circumcentre; and let O′ and A′ be the vertices opposite to O and A of the parallelograms of which BC is the common diagonal.

* Mr W. J. C. Miller in the *Lady's and Gentleman's Diary* for 1862, p. 74.

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Since $\angle BHC$ is supplementary to $\angle A$,
therefore $\angle BHC$ „ „ „ $\angle A'$;
therefore H is on the circumcircle of A'BC.
Now $\angle A'BH$ is right;
therefore A'H is a diameter of the circle A'BC;
therefore A'H passes through O' its centre.

SECOND DEMONSTRATION.*

Draw AA_1 parallel to BC;
join A'O and produce it to meet the circumcircle in R;
join AR meeting BC at X.

Then AR is perpendicular to BC,
and if we imagine the whole figure reflected in BC,
A_1 and O will reflect into the vertices of the two parallelograms on BC as diagonal.
Hence the line joining these vertices will meet AX at the point H, the reflection of R.
But since $\angle BHC = \angle BRC = 180° - \angle BAC$,
therefore H is the orthocentre of ABC;
therefore the straight line joining the two vertices of the parallelogram on BC as diameter passes through the orthocentre.

(41) *If through A, B, C there be drawn AC_1, BA_1, CB_1 making equal angles respectively with HA, HB, HC, a new triangle $A_1B_1C_1$ is formed, which is similar to ABC, and whose circumcentre† is H.*

FIGURE 49.

Because $\angle HCB_1 = \angle HBA_1$
therefore the points H, C, A_1, B are concyclic;
therefore $\angle A_1 = 180° - BHC = \angle A$.
Similarly $\angle B_1 = \angle B$, $\angle C_1 = \angle C$.

Join HB_1, HC_1.

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* "Conic" of St John's College, Cambridge, in the *Lady's and Gentleman's Diary* for 1863, p. 51.

† C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 34 (1825).

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Because $\angle ACH = \angle ABH$,

and $\angle ACH = \angle AB_1H$,

$\angle ABH = \angle AC_1H$;

therefore $\angle AB_1H = \angle AC_1H$;

therefore $HB_1 = HC_1$.

Similarly $HC_1 = HA_1$;

therefore H is the circumcentre of $A_1B_1C_1$.

(42) Since HA, HB, HC are respectively perpendicular to BC, CA, AB, the theorem of the preceding corollary is equivalent to the following:

If through the vertices of a triangle straight lines be drawn making equal angles with the opposite sides, they will form by their intersection a new triangle, which is similar to the original triangle, and which has for circumcentre the orthocentre of the original triangle.

A particular case of this theorem has already been given, that, namely, where the straight lines drawn through the vertices are parallel to the opposite sides. The triangle $A_1B_1C_1$ so formed, the anticomplementary triangle of ABC, is the maximum triangle that can be constructed under such conditions, and it it is equal to four times ABC.

(43) *Triangle XYZ is the triangle of minimum perimeter* inscribed in ABC.*

It is usually considered that this statement is proved † when it is shown that XY and XZ make equal angles with BC

„ YZ „ YX „ „ „ „ CA

„ ZX „ ZY „ „ „ „ AB.

No objection can be taken to the following proof ‡:

FIGURE 50.

Produce YZ both ways, making ZX_1 equal to ZX, YX_2 equal to YX; then X_1X_2 is the perimeter of XYZ.

Join BX_1, CX_2.

Because $\angle XZB = \angle AZY = \angle X_1ZB$

* J. F. de Tuschis a Fagnano in *Nova Acta Eruditorum anni* 1775, p. 296.

† See Prof. R. E. Allardice's paper "On a property of odd and even polygons" in the *Proceedings of the Edinburgh Mathematical Society*, VIII. 23 (1890).

‡ Marsano, *Considerazioni sul Triangolo Rettilineo*, pp. 18, 19 (1863).

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therefore triangles XZB and X_1ZB are congruent.

Similarly „ XYC „ X_2YC „ „ .

If now DEF be any other triangle inscribed in ABC, and along BX_1 there be taken BD_1 equal to BD, and along CX_2 there be taken CD_2 equal to CD, and FD_1, ED_2 be joined, it may be proved that $FD_1 = FD$, $ED_2 = ED$, and that consequently the line D_1FED_2 is the perimeter of DEF.

If D_1FED_2 is not straight, join D_1D_2 and join the vertex A with X, D, X_1, D_1, X_2, D_2.

Then $$AX_1 = AX = AX_2,$$
$$AD_1 = AD = AD_2;$$

therefore the triangles AX_1X_2, AD_1D_2 are isosceles.

And their vertical angles X_1AX_2, D_1AD_2 are equal,

since each is double of angle BAC;

therefore the triangles AX_1X_2, AD_1D_2 are similar.

Now AX_1 is less than AD_1, since AX is less than AD;

therefore X_1X_2 is less than D_1D_2, and consequently less than D_1FED_2.

If the triangle ABC be right-angled at A, the points Y, Z coalesce with A, X_1X_2 and D_1D_2 pass through A and are respectively double of AX and AD.

If the triangle ABC be obtuse-angled at A, the points Y, Z fall outside the triangle ABC (*Figure* 51) and X_1X_2 is now equal to $XY - YZ + ZX$. If therefore the preceding statements and proof are to hold good, the side YZ must be considered negative.

(44) *If XX_1, XX_2 be joined cutting AB, AC in P, Q, then PQ is the semiperimeter* of triangle XYZ.*

FIGURES 50, 51.

For P is the mid point of XX, and Q the mid point of XX_2;

therefore $PQ = \frac{1}{2}X_1X_2 =$ semiperimeter of XYZ.

P and Q are the feet of the perpendiculars from X on AB and AC.

If triangle ABC be obtuse-angled, the perimeter of XYZ must be understood with the qualification of the preceding corollary.

* Lhuilier, *Élémens d'Analyse*, p. 231 (1809). The proof in the text is given by Feuerbach, *Eigenschaften...des...Dreiecks*, Section VI., Theorem 8 (1822).

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(45) *If two triangles ABC, A'B'C' have their sides parallel, and one of them is circumscribed about and the other is inscribed in the same triangle DEF, the area of this last triangle is a mean proportional between the areas of the two others.**

FIGURE 52.

Let AB', AC' meet BC at P and Q. Through A' draw A'A'' parallel to B'C' or BC and meeting AC' at A''.

Join A''B', AA', B'Q.

Then $A'B'F = A'B'A$, $A'C'E = A'C'A$, $B'C'D = B'C'Q$;

therefore $DEF = AB'Q$, $A'B'C' = A''B'C'$.

Now $A''B'C' : AB'Q = A''C' : AQ$;

and $A''C' : AQ$ is the ratio of the altitudes of the similar triangles A'B'C', ABC.

Hence $A''C' : AQ = B'C' : BC$;

therefore $A''B'C' : AB'Q = B'C' : BC$.

Again $A\ B'Q : APQ = AB' : AP$
$= B'C' : PQ$;

and $A\ PQ : ABC = PQ : BC$;

therefore $A\ B'Q : ABC = B'C' : BC$;

therefore $A''B'C' : AB'Q = AB'Q : ABC$

or $A'B'C' : DEF = DEF : ABC$.

The terms *inscribed* and *circumscribed* have the following signification,

One triangle is inscribed in a second triangle when the vertices of the first are situated on the sides or the sides produced of the second; and in either case the second triangle is circumscribed about the first.

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* This theorem is due to Mr Rochat of Saint-Brieux, and is thus stated in Gergonne's *Annales de Mathématiques* II. 93 (1811-2).

If to any triangle T there be circumscribed another T', and to T' a third T'' having its sides respectively parallel to those of T; then to T'' a new triangle T''' having its sides respectively parallel to those of T', and so on: the triangles T, T', T'', T''' will be similar in pairs and form a geometrical progression.

The demonstration in the text is given by Mr Léon Anne, in the *Nouvelles Annales*, III. 27 (1844).

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(46) *If $I_1I_2I_3$ be the fundamental triangle, ABC its orthic triangle, and DEF the triangle formed by joining the points of contact of the incircle of ABC, then**

$$I_1I_2I_3 : ABC = ABC : DEF.$$

FIGURE 28.

In the same way if II_3I_2 be the fundamental triangle, ABC its orthic triangle, and $D_1E_1F_1$ the triangle formed by joining the points of contact of the first excircle of ABC, then

$$II_3I_2 : ABC = ABC : D_1E_1F_1;$$

and so on.

(47)
$$ABC : DEF = 2R : r$$
$$ABC : D_1E_1F_1 = 2R : r_1$$

and so on.

For
$$(ABC)^2 : (DEF)^2 = I_1I_2I_3 : DEF$$
$$= 4R^2 : r^2$$

since 2R and r are the radii of the circumcircles of the similar triangles $I_1I_2I_3$ and DEF.

(48) *If ABC be the fundamental triangle, DEF the triangle formed by joining the points of contact of the incircle of ABC, and X'Y'Z' the orthic triangle of DEF, then*

$$ABC : DEF = DEF : X'Y'Z'.$$

FIGURE 53.

For $\angle BDF = \angle DEF = \angle DY'Z'$;

therefore Y'Z' is parallel to BC.

Hence Z'X' ,, ,, CA

and X'Y' ,, ,, AB.

In the same way if ABC be the fundamental triangle, $D_1E_1F_1$ the triangle formed by joining the points of contact of the first excircle of ABC, and the orthic triangle of $D_1E_1F_1$ be constructed, it will be found that this orthic triangle has its sides parallel to those of ABC, and that $D_1E_1F_1$ is a mean proportional between it and ABC.

* The theorems (46)-(48) are given by Feuerbach, *Eigenschaften...des...Drciecks*, §§ 61, 8, 63 (1822).

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(49) Hence $I_1I_2I_3$, ABC, DEF, X'Y'Z', ...

are a series of triangles whose areas form a geometrical progression, the alternate terms being similar.

Other series may be obtained from

$$II_2I_3,\ ABC,\ D_1E_1F_1,\ \ldots, \text{ etc.}$$

(50) $$ABC : X'Y'Z' = 4R^2 : r^2.$$

DEF. If P be any point in the plane of ABC, and D, E, F be the projections of P on BC, CA, AB, then DEF is called the *pedal* triangle of P with respect to ABC.

(51) *If* H_1 , H_2 , H_3

be the orthocentres of the triangles

$$AEF\ ,\ BFD\ ,\ CDE$$

cut off from ABC by the sides of the pedal triangle DEF of any point P, the triangle $H_1H_2H_3$ is congruent and oppositely situated to DEF.

FIGURE 54.

Since PD, FH_2 are perpendicular to BC,
therefore PD is parallel to FH_2.
Similarly PF „ „ DH_2;
therefore PDH_2F is a parallelogram,
and $PD = FH_2$.
Hence also $PD = EH_3$,
therefore EFH_2H_3 is a parallelogram,
and $H_2H_3 = EF$.

FIGURE 55.

The sides of the four triangles

$$DEF,\ D_1E_1F_1,\ D_2E_2F_2,\ D_3E_3F_3$$

make with the sides of ABC the following

twelve triangles	whose orthocentres are
AEF , BFD , CDE	H_1, H_2 , H_3
AE_1F_1 , BF_1D_1 , CD_1E_1	H_1', H_2', H_2'
....	.....
.....	.....

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(52) The twelve orthocentres are situated in pairs on the six lines

$$II_1,\ II_2,\ II_3,\ I_2I_3,\ I_3I_1,\ I_1I_2.$$

(53) The four triangles

$$H_1H_2H_3, \quad H_1'H_2'H_3', \text{ and so on,}$$

are congruent and oppositely situated to

$$D\ E\ F, \quad D_1\ E_1\ F_1, \text{ and so on.}$$

(54) The following figures are rhombi :

$$\begin{array}{lll} D\,H_3\,E\,I\,, & E\,H_1\,F\,I\,, & F\,H_2\,D\,I \\ D_1H_3'E_1I_1\,, & E_1H_1'F_1I_1\,, & F_1H_2'D_1I_1 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{array}$$

their sides being r, r_1, r_2, r_3 respectively.

(55) The following figures are equilateral hexagons :

$$DH_3EH_1FH_2,\ D_1H_3'E_1H_1'F_1H_2',\ \ldots\ldots\ldots,\ \ldots\ldots\ldots$$

their perimeters being $6r$, $6r_1$, $6r_2$, $6r_3$ respectively.

(56) $I\ ,\quad I_1\ ,\quad I_2\ ,\quad I_3$

which are the circumcentres of the triangles

$$D\ E\ F, \quad D_1\ E_1\ F_1, \quad \textit{and so on,}$$

*are the orthocentres of the triangles**

$$H_1H_2H_3, \quad H_1'H_2'H_3', \quad \textit{and so on.}$$

Take, for example, the triangle $H_1H_2H_3$.

Because H_1I is perpendicular to $E\,F$,

therefore H_1I ,, ,, ,, H_2H_3.

Similarly for H_2I and H_3I.

(57) *If* $H_0\ ,\quad H_0'\ ,\quad H_0''\ ,\quad H_0'''$

be the orthocentres of the triangles

$$D\ E\ F, \quad D_1\ E_1\ F_1, \quad \textit{and so on,}$$

*they will be the circumcentres of the triangles**

$$H_1H_2H_3, \quad H_1'H_2'H_3', \quad \textit{and so on.}$$

---

* The first parts of (56) and (57) are given by Feuerbach, *Eigenschaften...des... Dreiecks*, §§ 87, 88 (1822).

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FIGURE 56.

Take, for example, the triangle $H_1H_2H_3$.

Because DH_0 is perpendicular to EF,
therefore DH_0 ,, ,, H_2H_3.
And since $DH_2 = DH_3$,
therefore DH_0 bisects H_2H_3
therefore DH_0 passes through the circumcentre of $H_1H_2H_3$

Similarly for EH_0 and FH_0.

(58) $$H_0H_1 = H_0H_2 = H_0H_3 = I\ D = I\ E = I\ F = r$$

For H_0 and I are the circumcentres of two congruent triangles $H_1H_2H_3$ and DEF.

Similarly for H_0', I_1, and so on.

(59) The following figures are parallelograms:

$$DIH_1H_0,\ EIH_2H_0,\ FIH_3H_0;$$

they have a common diagonal IH_0;
their other diagonals intersect at the mid point of IH_0.

Similarly for I_1H_0', and so on.

(60) The homothetic centre of DEF, $H_1H_2H_3$ is the mid point of IH_0.

Similarly for $D_1E_1F_1$, $H_1'H_2'H_3'$, and so on.

(61) The following figures are rhombi:

$$DH_3H_0H_2,\ EH_1H_0H_3,\ FH_2H_0H_1,$$

and their sides are equal to r.

Three other triads of rhombi can be obtained by putting subscripts and accents to the preceding letters.

(62) If from H_1, H_2, H_3 perpendiculars be drawn to BC, CA, AB, these perpendiculars will be concurrent at H_0.

Since $EH_1H_0H_3$ is a rhombus,
therefore H_1H_0 is parallel to EH_3
therefore H_1H_0 is perpendicular to BC.

Similarly for H_2H_0 and H_3H_0.

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(63) Since I and H_0 are the circumcentre and orthocentre of DEF and the orthocentre and circumcentre of $H_1H_2H_3$, these two triangles have the same nine-point circle,* and its centre is the mid point of IH_0.

FIGURE 56.

(64) In triangle DEF

$$\left.\begin{matrix} D\,I, & D\,H_0 \\ E\,I, & E\,H_0 \\ F\,I, & F\,H_0 \end{matrix}\right\} \text{ are isogonals with respect to } \left\{\begin{matrix} D \\ E \\ F \end{matrix}\right.$$

(65) In triangle $H_1H_2H_3$

$$\left.\begin{matrix} H_1I, & H_1H_0 \\ H_2I, & H_2H_0 \\ H_3I, & H_3H_0 \end{matrix}\right\} \text{ are isogonals with respect to } \left\{\begin{matrix} H_1 \\ H_2 \\ H_3 \end{matrix}\right.$$

(66) *Of the perpendiculars to BC from H_1, H_2, H_3, I the first is equal to the sum of the other three.*†

FIGURE 57.

Let the feet of the perpendiculars on BC from H_1, H_2, H_3 be X_1, X_2, X_3;
let IH_1 meet EF at D';
from D' draw a perpendicular to BC, meeting BC at D'' and H_2H_3 at L.

Then D' is the mid point of EF and IH_1;
therefore D'' ,, ,, ,, ,, ,, X_2X_3 ,, X_1D.
Also L ,, ,, ,, ,, ,, H_2H_3;
therefore L ,, ,, ,, ,, ,, DH_0,
since $DH_3H_0H_2$ is a rhombus.
Hence $H_2X_2 + H_3X_3 = 2LD'' = H_0X_1$;
and $ID = H_0H_1$;
therefore $H_2X_2 + H_3X_3 + ID = H_1X_1$.

* Feuerbach, § 89.

† Feuerbach, § 80. The mode of proof is not his.

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In triangle ABC, the perpendiculars AX, BY, CZ intersect at H the orthocentre, and XYZ is the orthic triangle.

FIGURE 58.

This figure has reference to the properties (67)–(82). The reader would find it convenient if he constructed a copy of it on a large scale.

Of the triangles	let the orthocentres be
AYZ, XBZ, XYC	H_1 , H_2 , H_3
HYZ, XCZ, XYB	H_1' , H_2' , H_3'
CYZ, XHZ, XYA	H_1'', H_2'', H_3''
BYZ, XAZ, XYH	H_1''', H_2''', H_3'''

(67) Of these H points, four pairs are collinear with X, four with Y, and four with Z, that is, through

X	pass	H_2H_2''',	H_3H_3'',	$H_2'H_2''$, $H_3'H_3'''$
Y	,,	H_3H_3',	H_1H_1''',	$H_3''H_3'''$, $H_1'H_1''$
Z	,,	H_1H_1'',	H_2H_2',	$H_1'H_1'''$, $H_2''H_2'''$

(68) The four* triangles $H_1H_2H_3$, $H_1'H_2'H_3'$ etc., are congruent and oppositely situated to XYZ.

(69) *The three triangles* $H_1'H_1''H_1'''$, $H_2'H_2''H_2'''$, $H_3'H_3''H_3'''$ *are congruent and oppositely situated to ABC; and* H_1, H_2, H_3 *are their respective orthocentres.*

Take for example $H_3'H_3''H_3'''$.

Because $H_3''H_3'''$ passes through Y and is perpendicular to AX,
therefore $H_3''H_3'''$ is parallel to BC.
Similarly $H_3'''H_3'$ is parallel to CA.
Again $H_2'H_3'$ is equal and parallel to $H_2''H_3''$;
therefore $H_3'H_3''$,, ,, ,, ,, ,, $H_2'H_2''$.
Now $H_2'H_2''$ passes through X and is perpendicular to CZ;
therefore $H_3'H_3''$ is parallel to AB.
Hence $H_3'H_3''H_3'''$ is similar and oppositely situated to ABC.

* Feuerbach (§ 90) proves the congruency of XYZ, $H_1H_2H_3$

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Because $H_3'H_3$ passes through Y and is perpendicular to BC,
and $H_3''H_3$ „ „ X „ „ „ „ CA;
therefore Y, X are the feet of two of the perpendiculars,
and H_3 is the orthocentre of triangle $H_3'H_3''H_3'''$.

Lastly, since H, X in triangle ABC
correspond to H_3, Y „ „ $H_1'H_2''H_3'''$, and $HX = H_3Y$,
therefore triangles ABC, $H_1'H_2''H_3'''$ are congruent.

(70) *If* $H_1'H_1$ *meet* $H_1''H_1'''$ *at* X_1,
$H_2''H_2$ „ $H_2''H_2'$ „ Y_1,
$H_3'''H_3$ „ $H'H_3'$ „ Z_1:

then the feet of the perpendiculars of triangle

$H_1'H_1''H_1'''$ *are* X_1, Z, Y,
$H_2'H_2''H_2'''$ „ Z, Y_1, X,
$H_3'H_3''H_3''$ „ Y, X, Z_1;

and the sides of triangle $X_1Y_1Z_1$ ***pass through*** X, Y, Z ***and are there bisected.***

Because triangles $H_1'H_2'H_3'''$ and ABC are congruent and oppositely situated,
therefore their orthic triangles X_1ZY and XYZ are congruent and oppositely situated.
Similarly ZY_1X and YXZ_1 are congruent and oppositely situated to XYZ;
therefore Y_1Z_1 passes through X and is bisected at X
Z_1X_1 „ „ Y „ „ „ „ Y
X_1Y_1 „ „ Z „ „ „ „ Z.

(71) *ABC,* $X_1Y_1Z_1$ *have the same nine-point circle.*

For X, Y, Z, the feet of the perpendiculars of ABC, are the mid points of the sides of $X_1Y_1Z_1$.

(72) *If* O, O_1, O_2, O_3 ***be the circumcentres of***

ABC, HCB, CHA, BAH:

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then the point of concurrency of

AH_1 ,	BH_2 ,	CH_3	*is*	O
HH_1' ,	CH_2' ,	BH_3'	,,	O_a
CH_1'' ,	HH_2'' ,	AH_3''	,,	O_b
BH_1''',	AH_2''',	HH_3'''	,,	O_c

and O , O_a , O_b , O_c *are orthocentres of* $H_1H_2H_3$, $H_1'H_2'H_3'$, $H_1''H_2''H_3''$, $H_1'''H_2'''H_3'''$.

For AH_1, BH_2, CH_3 are respectively perpendicular to YZ, ZX, XY;

and their concurrency is established by Steiner's theorem concerning orthologous triangles. See § 6 (1).

Since AH_1. BH_2, CH_3 are respectively perpendicular to YZ, ZX, XY, they are therefore perpendicular to H_2H_3, H_3H_1, H_1H_2,

and consequently concurrent at the orthocentre of $H_1H_2H_3$.

(73) *If the homothetic centre of the triangles*

XYZ	*and*	H_1 H_2 H_3	*be*	T
XYZ	,,	H_1' H_2' H_3'	,,	T_1
XYZ	,,	H_1'' H_2'' H_3''	,,	T_2
XYZ	,,	H_1''' H_2''' H_3'''	,,	T_3 ,

then $T_1T_2T_3$ *is similar and oppositely situated to* ABC, *and* T, T_1, T_2, T_3 *form an orthic tetrastigm.*

For T_2 is the mid point of XH_1''

T_3 ,, ,, ,, ,, ,, XH_1''';

therefore T_2T_3 is parallel to $H_1''H_1'''$ and equal to half of it;

therefore T_2T_3 ,, ,, ,, B C ,, ,, ,, ,, ,, ,, .

Again T is the mid point of XH_1

T_1 ,, ,, ,, ,, ,, XH_1';

therefore TT_1 is parallel to H_1H_1' and equal to half of it;

therefore TT_1 is perpendicular to $H_1''H_1'''$ or to T_2T_3,

and T is orthocentre of $T_1T_2T_3$.

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(74) *The point T is the centre of the three parallelograms*

$$YZH_2H_3.\ ZXH_3H_1,\ XYH_1H_2.$$

For YH_2, ZH_3 intersect at T.

Similarly T_1, T_2, T_3 are each the centre of three parallelograms.

(75) *If X', Y', Z' be the mid points of YZ, ZX, XY, then the point of concurrency of*

$H_1\ X'$,	$H_2\ Y'$,	$H_3\ Z'$	*is*	H
$H_1'\ X'$,	$H_2'\ Y'$,	$H_3'\ Z'$	,,	A
$H_1''\ X'$,	$H_2''\ Y'$,	$H_3''\ Z'$	,,	B
$H_1'''X'$,	$H_2'''Y'$,	$H_3'''Z'$	,,	C.

For YH_1 and HZ are parallel, and so are ZH_1 and HY;
therefore HYH_1Z is a parallelogram;
therefore HH_1 and YZ bisect each other,
that is, H_1X' passes through H.

Again, ABC, $H_1'H_1''H_1'''$ are congruent and oppositely situated, and Y in ABC corresponds to Z in $H_1'H_1''H_1'''$;
therefore $AYH_1'Z$ is a parallelogram;
therefore AH_1' and YZ bisect each other,
that is, $H_1'X'$ passes through A.

(76) Let the incircle and excircles of XYZ be denoted by their centres H, A, B, C; then the radical axes of

	H, A;	H, B;	H, C;	B, C;	C, A;	A, B
are	$T_2\ T_3$,	$T_3\ T_1$,	$T_1\ T_2$,	$T_1\ T$,	$T_2\ T$,	$T_3\ T$.

For T_2T_3 is perpendicular to HA, and bisects YZ.

(77) The circles A, B, C; H, C, B; C, H, A; B, A, H have T, T_1, T_2, T_3 for radical centres.

(78) *X', Y' Z' are the feet of the perpendiculars of $T_1T_2T_3$.*

For in triangle AXH_1' the mid point of XH_1' is T_1,
and T_1T is parallel to AX;
therefore T_1T passes through X', the mid point of AH_1'.

Hence T, T_1, T_2, T_3 are the incentre and the excentres of the triangle X'Y'Z'. Compare § 5, (11), (12).

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(79) The homothetic centre of the triangles

$T_1T_2T_3$ and $H_1'H_1''H_1'''$ is X
$T_1T_2T_3$ „ $H_2'H_2''H_2'''$ „ Y
$T_1T_2T_3$ „ $H_3'H_3''H_3'''$ „ Z.

For T_2, T_3 are mid points of XH_1'', XH_1'''.

(80) Since X'Y'Z' is the complementary triangle of XYZ, and T, T_1, T_2, T_3 are the incentre and excentres of X'Y'Z', and H, A, B, C „ „ „ „ „ „ XYZ; therefore HT, AT_1, BT_2, CT_3 all pass through the centroid of XYZ. See § 2.

If G' denote this centroid *

then $HG' : TG' = AG' : T_1G' = BG' : T_2G' = CG' : T_3G'$
$= 2 : 1.$

(81) Since H is the incentre, G' the centroid, of XYZ, and T the incentre of X'Y'Z', if HG'T be produced to J' so that TJ' = HT, then J' will be the incentre of $X_1Y_1Z_1$.

Similarly J_1', J_2', J_3', situated on AT_1, BT_2, CT_3, so that $T_1J_1' = AT_1$ and so on, will be the first, second, and third excentres of $X_1Y_1Z_1$.

These statements follow from the first few corollaries of § 2.

(82) The tetrads of points

H, G', T, J'; A, G', T_1, J_1'; B, G', T_2, J_2'; C, G', T_3, J_3'

form harmonic ranges.

(83) Since triangles $I_1I_2I_3$, ABC stand to each other in the same relation as ABC, XYZ, the second being the orthic triangle of the first, it may be convenient to state in another form some of the results already established.

The means of transliteration from the one form to the other will be afforded by the following lists of corresponding points.

* G' would naturally denote the centroid of triangle A'B'C', but G is the centroid both of ABC and A'B'C'.

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$A, B, C, H, X, Y, Z, H_1, H_2, H_3$

correspond to

$I_1, I_2, I_3, I, A, B, C, H_a, H_b, H_c$

and

$O, O_a, O_b, O_c, G', T, T_1, T_2, T_3$

correspond to

$O_0, O_1, O_2, O_3, G, L, L_1, L_2, L_3$

and

$X', Y', Z', X_1, Y_1, Z_1, J', J_1', J_2', J_3'$

correspond to

$A', B', C', A_1, B_1, C_1, J, J_1, J_2, J_3$.

Hence the following results* are obtained :

(*a*) The orthocentres of the triangles BCI_1, CAI_2, ABI_3 form the vertices of a triangle $H_aH_bH_c$ which is congruent to the fundamental triangle ABC, and has its sides parallel to the corresponding sides of ABC.

(*b*) AH_a, BH_b, CH_c are concurrent at the radical centre of I_1, I_2, I_3.

(*c*) The radical centre bisects AH_a, BH_b, CH_c.

(*d*) The radical axes of the I circles bisect the sides of $H_aH_bH_c$.

(*e*) O_0 is the orthocentre of $H_aH_bH_c$.

(*f*) AD_1, BE_2, CF_3 are concurrent at J the incentre of $H_aH_bH_c$. The points J, I, L are collinear.

(*g*) H_a, A', I ; H_b, B', I ; H_c, C', I are collinear.

(*h*) A', B', C' bisect H_aI, H_bI, H_cI.

FIGURE 59.

In triangle ABC, the points H, X, Y, Z

are the orthocentre and feet of the perpendiculars ;

the various I, D, E, F points are the centres and points of contact of the incircle and the excircles.

The rest of the notation will be explained as it is wanted.

* See Professor Johann Döttl's *Neue merkwürdige Punkte des Dreiecks*, pp. 40-46 (no date). The proofs given in this noteworthy pamphlet are analytical.

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(84) The perpendicular AX contains the intersection of

D_2E_2,	D_3F_3	namely	X_0
D_3E_3,	D_2F_2	,,	X_1
D E ,	D_1F_1	,,	X_2
D_1E_1,	D F	,,	X_3.

The perpendicular BY contains the intersection of

E_3F_3,	E_1D_1	namely	Y_0
E_2F_2,	E D	,,	Y_1
E_1F_1,	E_3D_3	,,	Y_2
E F ,	E_2D_2	,,	Y_3.

The perpendicular CZ contains the intersection of

F_1D_1,	F_2E_2	namely	Z_0
F D ,	F_3E_3	,,	Z_1
F_3D_3,	F E	,,	Z_2
F_2D_2,	F_1E_1	,,	Z_3.

FIGURE 60.

Through A draw a parallel to BC;

let D_2E_2 meet AX at X_0 and the parallel at S.

Then triangles CD_2E_2, ASE_2 are similar;

and because $CD_2 = CE_2$

therefore $AS = AE_2 = s_3$.

Now triangles AX_0S, DIC have their sides respectively parallel to each other; therefore they are similar.

But $AS = s_3 = DC$;

therefore $AX_0 = DI = r$.

Again if D_3F_3 meet the parallel through A at T,

and AX at X_0', it may be proved that

$$AT = AF_3 = s_2 = DB$$

and that triangles $AX_0'T$, DIB are congruent;

therefore $AX_0' = DI = r$,

and X_0, X_0' are the same point.

The other properties are proved in a manner exactly analogous.

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(85) FIGURE 59.

$$AX_0 = BY_0 = CZ_0 = r$$
$$AX_1 = BY_1 = CZ_1 = r_1$$
$$AX_2 = BY_2 = CZ_2 = r_2$$
$$AX_3 = BY_3 = CZ_3 = r_3.$$

Attention may be directed here and later on to the way in which the various suffixes occur.

The triads of lines

$$\left.\begin{array}{lll} E_1F_1, & F_2D_2, & D_3E_3 \\ EF, & F_3D_3, & D_2E_2 \\ E_3F_3, & FD, & D_1E_1 \\ E_2F_2, & F_1D_1, & DE \end{array}\right\}$$

determine the triangles

$$\left\{\begin{array}{l} X_1Y_2Z_3 \\ X_0Y_3Z_2 \\ X_3Y_0Z_1 \\ X_2Y_1Z_0 \end{array}\right.$$

(86) The four triangles

$$X_1Y_2Z_3, \quad X_0Y_3Z_2, \quad X_3Y_0Z_1, \quad X_2Y_1Z_0$$

are respectively similar and oppositely situated to

$$I_1\ I_2\ I_3, \quad I\ I_3\ I_2, \quad I_3\ I\ I_1, \quad I_2\ I_1\ I$$

and H, the orthocentre of ABC, is the circumcentre of the four.

Since Y_2Z_3 is perpendicular to AI_1,
therefore Y_2Z_3 is parallel to I_2I_3.
Similarly for Z_3X_1 and X_1Y_2.

Again $\angle HY_2Z_3 = \angle CAI_1$
because the sides of the one are perpendicular to those of the other;
and $\angle HZ_3Y_2 = \angle BAI_1$, for a similar reason;
therefore $\angle HY_2Z_3 = \angle HZ_3Y_2$;
therefore $HY_2 = HZ_3$.
Similarly $HZ_3 = HX_1$;
therefore H is the circumcentre of $X_1Y_2Z_3$.

(87) The radii of the circumcircles of

$$X_1Y_2Z_3, \quad X_0Y_3Z_2, \quad X_3Y_0Z_1, \quad X_2Y_1Z_0$$

are $2R + r$, $2R - r_1$, $2R - r_2$, $2R - r_3$.

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For
$$\begin{aligned} & HX_1 + HY_2 + HZ_3 \\ &= AX_1 + BY_2 + CZ_3 + HA + HB + HC \\ &= \quad r_1 + \quad r_2 + \quad r_3 + 2(k_1 + k_2 + k_3) \\ &= 4R + \quad r + 2r + 2R \\ &= 6R + 3r. \end{aligned}$$

(88) $X_0D = AI$, $X_1D_1 = AI_1$, $X_2D_2 = AI_2$, $X_3D_3 = AI_3$
$Y_0E = BI$, $Y_1E_1 = BI_1$, $Y_2E_2 = BI_2$, $Y_3E_3 = BI_3$
$Z_0F = CI$, $Z_1F_1 = CI_1$, $Z_2F_2 = CI_2$, $Z_3F_3 = CI_3$

Because AX_0 is equal and parallel to ID,
therefore $AIDX_0$ is a parallelogram;
therefore $X_0D = AI$.

(89) In the four pairs of triangles

$X_1Y_2Z_3$, $X_0Y_3Z_2$, $X_3Y_0Z_1$, $X_2Y_1Z_0$
$I_1I_2I_3$, $I\ I_3I_2$, $I_3I\ I_1$, $I_2\ I_1I$

consider the intersections of the sides.*

Y_2Z_3	intersects	I_1I_2,	I_1I_3	at	V_1, W_1
Z_3X_1	,,	I_2I_3,	I_2I_1	,,	W_2, U_2
X_1Y_2	,,	I_3I_1,	I_3I_2	,,	U_3, V_3
Y_3Z_2	,,	$I\ I_3$,	$I\ I_2$	,,	V', W'
Z_2X_0	,,	I_3I_2,	I_3I	,,	W_2, U''
X_0Y_3	,,	I_2I ,	I_2I_3	,,	U''', V_3
Y_0Z_1	,,	I_3I ,	I_3I_1	,,	V', W_1
Z_1X_3	,,	$I\ I_1$,	$I\ I_3$	,,	W'', U''
X_3Y_0	,,	I_1I_3,	I_1I	,,	U_3, V'''
Y_1Z_0	,,	I_2I_1,	I_2I	,,	V_1, W'
Z_0X_2	,,	I_1I ,	I_1I_2	,,	W'', U_2
X_2Y_1	,,	$I\ I_2$,	$I\ I_1$	,,	U''', V'''.

It will be seen that several theorems are embedded in the preceding notation.

* The notation here is somewhat complicated, but it could not well be otherwise. I have made various attempts to simplify it, but with little success; what is gained in one respect is lost in another.

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(90) The sides of the four triangles

$$DEF,\quad D_1E_1F_1,\quad D_2E_2F_2,\quad D_3E_3F_3$$

contain each four other points of the diagram.

E F	contains	Y_3	Z_2	V'	W'
F D	,,	Z_1	X_3	W''	U''
D E	,,	X_2	Y_1	U'''	V'''
E_1F_1	,,	Y_2	Z_3	V_1	W_1
F_1D_1	,,	Z_0	X_2	W'''	U_2
D_1E_1	,,	X_3	Y_0	U_3	V'''
E_2F_2	,,	Y_1	Z_0	V_1	W'
F_2D_2	,,	Z_3	X_1	W_2	U_2
D_2E_2	,,	X_0	Y_3	U'''	V_3
E_3F_3	,,	Y_0	Z_1	V'	W_1
F_3D_3	,,	Z_2	X_0	W_2	U''
D_3E_3	,,	X_1	Y_2	U_3	V_3

(91) The twelve EF, FD, DE lines determine, by their intersections with the six lines of the orthic tetrastigm $II_1I_2I_3$, pairs of feet of the perpendiculars of the triangles

I_1BC,	AI_2C,	ABI_3
IBC,	AI_3C,	ABI_2
I_3BC,	AIC,	ABI_1
I_2BC,	AI_1C,	ABI.

The other twelve feet are the various D, E, F points.

It may be useful to remember that these four triads of triangles are similar to

$$I_1I_2I_3,\quad II_3I_2,\quad I_3II_1,\quad I_2I_1I.$$

The following proof of one of these properties may be sufficient:

Because $CD_1 = CE_1$

therefore triangles CD_1V_1, CE_1V_1 are congruent

and $$\angle CD_1V_1 = \angle CE_1V_1 = \tfrac{1}{2}(B + C)$$
$$= \angle I_3AB.$$

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Now triangle I_1BC is similar to $I_1I_2I_3$,
the sides I_1B, I_1C, BC being homologous to
„ „ I_1I_2, I_1I_3, I_2I_3
and I_1D_1 being homologous to I_1A,
so that D_1 and A are homologous points;
therefore V_1 „ B „ „ „ ,
since $\angle CD_1V_1 = \angle I_3AB$.
But B is the foot of the perpendicular on I_1I_3 from I_2;
therefore V_1 „ „ „ „ „ „ „ I_1C „ B.

(92) The following quartets of points form orthic tetrastigms:

$$X_0X_1D_2D_3\ ;\quad Y_0Y_2E_3E_1\ ;\quad Z_0Z_3F_1F_2$$
$$X_2X_3D\ D_1\ ;\quad Y_3Y_1E\ E_2\ ;\quad Z_1Z_2F\ F_3\ .$$

(93) Through the mid point of

BC	pass	X_0I,	X_1I_1,	X_2I_2,	X_3I_3
CA	„	Y_0I,	Y_1I_1,	Y_2I_2,	Y_3I_3
AB	„	Z_0I,	Z_1I_1,	Z_2I_2,	Z_3I_3.

Take* for example X_1I_1.

Triangles $X_1D_2D_3$, I_1BC are similar and oppositely situated;
therefore X_1I_1 passes through their centre of similitude.
But D_3C „ „ „ „ „ „ ;
therefore if X_1I_1 cut D_3C at A', the centre of similitude is A'.

Hence
$$\frac{A'B}{A'C} = \frac{A'D_2}{A'D_3}$$
$$= \frac{A'B + A'D_2}{A'C + A'D}$$
$$= \frac{BD_2}{CD_3} = \frac{s}{s}\ ;$$

therefore A' is the mid point of BC.

A shorter demonstration of this would be obtained if (95) were proved before (93).

* This method of proof is due to Professor Neuberg.

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For $X_1U_3I_1U_2$ is a parallelogram :

therefore X_1I_1 bisects U_3U_2 ;

therefore „ „ B C .

(94) The four centres of homology of the four pairs of triangles $X_1Y_2Z_3$, $I_1I_2I_3$, and so on, are the symmedian points of these pairs of triangles.

For I_1X_1 bisects BC,

and BC is antiparallel to I_2I_3 with respect to $\angle I_1$;

therefore I_1X_1 is a symmedian of $I_1I_2I_3$.

Since X_1I_1 bisects BC, it must also bisect D_2D_3.

Now D_2D_3 is antiparallel to Y_2Z_3 with respect to $\angle X_1$;

therefore X_1I_1 is a symmedian of $X_1Y_2Z_3$.

(95) All the U points are on a line parallel to BC

„ „ V „ „ „ „ „ „ „ CA

„ „ W „ „ „ „ „ „ „ AB.

(96) $U_2U_3 = V_3V_1 = W_1W_2 = s$

$U''U''' = V_3V' = W'W_2 = s_1$

$U''U_3 = V'''V' = W_1W'' = s_2$

$U_2U''' = V'''V_1 = W'W'' = s_3$.

Because D_2U_2 is parallel to BU_3,

and CU_2 „ „ „ D_3U_3,

and $CD_2 = s_1 = BD_3$;

therefore $D_2U_2 = BU_3$;

therefore $U_2U_3BD_2$ is a parallelogram ;

therefore U_2U_3 is parallel and equal to BD_2, that is to s.

Similarly the other UU lines are parallel to BC ;

therefore the U points are collinear.

The U points lie on the line B'C'.

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(97) The following sets of six points are concyclic

$$U_2 , \quad U_3 , \quad V_3 , \quad V_1 , \quad W_1 , \quad W_2$$
$$U'' , \quad U''' , \quad V_3 , \quad V' , \quad W' , \quad W_2$$
$$U'' , \quad U_3 , \quad V''' , \quad V' , \quad W_1 , \quad W''$$
$$U_2 , \quad U''' , \quad V''' , \quad V_1 , \quad W' , \quad W''.$$

Because D_2V_3, D_3W_2 are two of the perpendiculars of $X_1D_2D_3$;

therefore W_2V_3 is antiparallel to D_2D_3 with respect to $\angle X_1$;

therefore W_2V_3 is antiparallel to U_2U_3;

therefore W_2, V_3, U_3, U_2 are concyclic.

Similarly U_3, W_1, V_1, V_3 ,, ,,

and V_1, U_2, W_2, W_1 ,, ,,

Hence all the six points are concyclic.

The four circles are the Taylor circles of the orthic tetrastigm

$$II_1I_2I_3.$$

(98) If the centres of these circles be denoted by

$$O_0 , \quad O_1 , \quad O_2 , \quad O_3$$

then these four points form an orthic tetrastigm.

They are the incentre and the excentres of the complementary triangle A'B'C'.

(99) The six II lines of the orthic tetrastigm $II_1I_2I_3$ are the radical axes of the circles O_0, O_1, O_2, O_3 taken in pairs; and the four I points of the same tetrastigm are the radical centres of the circles O_0, O_1, O_2, O_3 taken in threes.

(100) The following are symmetrical trapeziums:

$$W_2V_3 \; W_1V_1 ; \quad U_3 \; W_1U_2W_2 ; \quad V_1U_2V_3 \; U_3 ;$$
$$W_2V_3 \; W'V' ; \quad U'''W'U''W_2 ; \quad V'U''V_3 \; U''' ;$$
$$W''V'''W_1V' ; \quad U_3 \; W_1U''W'' ; \quad V'U''V'''U_3 ;$$
$$W''V'''W'V_1 ; \quad U'''W'U_2W'' ; \quad V_1U_2V'''U'''.$$

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Professor Fuhrmann gives the following property, but his proof is too long for insertion here:

The axis of homology of the triangles

$$ABC \text{ and } X_1Y_2Z_3$$

is perpendicular to HI.

[illegible] properties (84), (85), (91) are given by W. H. Levy of [illegible] *Leybourn's Mathematical Repository's Diary* for 1857, pp. 50-1, in his answer [illegible] the previous year.

[illegible] *Nouvelles Annales de Mathématiques* (Paris, 1873) the following [illegible]:

Three [illegible] of a triangle ABC which are situated [illegible] a new triangle A'B'C' is formed. (1) *Find [illegible] P such that AA', BB', CC' are the altitudes of ABC.* [illegible] *circumcircle of A'B'C'.*

[illegible] *Mathématique*, I. 50-3 (1874), Professor [illegible] the question, in which (confining himself [illegible] (91), (92), (93), (94) and one or two [illegible] Professor N[illegible] 6th edition of Casey's *Sequel to Euclid*, [illegible] Professor [illegible] in his *Synthetische Beweise planimetrischer* [illegible]

[illegible] communicated to the Edinburgh Mathematical [illegible]

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§ 6. Euler's Line.

*The circumcentre, the centroid, and the orthocentre of a triangle are collinear, and the distance between the first two is half the distance between the last two.**

Figure 61.

First Demonstration.†

Let A′, B′ be the mid points of BC, CA, and let the perpendiculars to BC, CA at A′, B′ meet at O;

then O is the circumcentre.

Let the perpendiculars AX, BY meet at H;

then H is the orthocentre.

Join OH and let AA′ meet it at G. Join A′B′.

Because triangles HAB, OA′B′ have their sides respectively parallel to each other, they are similar;

therefore $HA : OA' = AB : A'B' = 2 : 1$.

Again triangles HAG, OA′G are similar;

therefore $HG : OG = HA : OA' = 2 : 1$

that is, AA′ cuts OH so that HG = twice OG.

Hence also the medians from B and C cut OH

so that HG = twice OG;

therefore G is the centroid, and H, G, O are collinear.

This is also a proof that the medians are concurrent.

Second Demonstration.

Let AA′ be the median from A, G the centroid, and O the circumcentre.

Join OA′, OG, and let AX the perpendicular from A to BC meet OG produced at H.

---

* Proved by Euler in 1765. His proof will be found in *Novi Commentarii Academiae ... Petropolitanae*, XI. 114. An abstract of this paper of Euler's is printed in the *Proceedings of the Edinburgh Mathematical Society*, IV. 51-55 (1886).

† This method of proof is given in Carnot's *Géométrie de Position*, § 131 (1803). The second and third methods are imitations of it.

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Then triangles HAG, OA'G are similar;

therefore $HG : OG = AG : A'G = 2 : 1$,

that is, AX cuts OG produced so that HG = twice OG.

Hence also the perpendiculars from B and C cut OG produced so that HG = twice OG;

therefore H is the orthocentre, and H, G, O are collinear.

This is also a proof that the perpendiculars to the sides from the vertices are concurrent.

Third Demonstration.

Let AX be the perpendicular, AA' the median, from A to BC; and let H be the orthocentre, G the centroid.

Join HG, and let the perpendicular from A' to BC meet HG produced at O.

Then triangles HAG, OA'G are similar;

therefore $HG : OG = AG : A'G = 2 : 1$,

that is, the perpendicular to BC from its mid point cuts HG produced so that HG = twice OG.

Hence also the perpendiculars to CA, AB from their mid points cut HG produced so that HG = twice OG;

therefore O is the circumcentre, and H, G, O are collinear.

This is also a proof that the perpendiculars to the sides from their mid points are concurrent.

Fourth Demonstration.*

Figure 62.

Let H be the orthocentre, determined by drawing AX, BY perpendicular to BC, CA; O the circumcentre, determined by drawing A'O, B'O perpendicular to BC, CA from their mid points A', B'. Join HO and let it meet the median AA' at G.

Bisect HA, HB at U, V, and GA, GH at P, Q;

join UV, PQ, A'B'.

* This mode of proof assumes only the first book of Euclid's *Elements* and its immediate consequences.

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Then $A'B'$ is parallel to AB and equal to $\frac{1}{2}AB$,
and $U\,V$ is parallel to AB and equal to $\frac{1}{2}AB$;
therefore $A'B'$ is parallel to UV and equal to UV.
Because $O\,A'$ and HU are both perpendicular to BC;
therefore $O\,A'$ is parallel to HU.
Similarly $O\,B'$ is parallel to HV.
Hence the triangles $OA'B'$, HUV are mutually equiangular,
and, since $A'B' = UV$, congruent.
Therefore $$OA' = HU = \tfrac{1}{2}AH.$$

Again PQ is parallel to AH and equal to $\frac{1}{2}AH$;
therefore PQ is parallel to OA' and equal to OA'.
Hence the triangles $A'GO$, PGQ are congruent;
therefore $A'G = PG = \frac{1}{2}AG$;
therefore G is the centroid, and $OG = QG = \frac{1}{2}HG$.

The straight line HGO is frequently called *Euler's line.*

(1) *The twelve radii drawn from the incentre and the excentres of a triangle perpendicular to the sides of the triangle meet by threes in four points, and these four points are the circumcentres of the triangles* *

$$I_1I_2I_3,\quad II_3I_2,\quad I_3II_1,\quad I_2I_1I.$$

FIGURE 63.

The triads of concurrent radii are

I_1D_1, I_2E_2, I_3F_3	$I\,D$, I_3E_3, I_2F_2
I_3D_3, $I\,E$, I_1F_1	I_2D_2, I_1E_1, $I\,F$

and the theorem follows at once from the converse of the first part of § 5, (32).

A second proof of the concurrency of these four triads may be derived from Oppel's theorem in § 2 and the expressions in § 4, (5).

---

* The results (1)–(7) are given by T. S. Davies in the *Philosophical Magazine*, II. 26-34 (1827). The concurrency of the first triad at the circumcentre of triangle $I_1I_2I_3$, and the length of the radius, $2R$, of that triangle were pointed out by Benjamin Bevan in Leybourn's *Mathematical Repository*, new series, I. 18 (pagination of questions), 143 (pagination of Part I.) in 1804. Compare the subscripts in the designations of the four I triangles with the subscripts of the radii which meet at their circumcentres.

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A third proof may be got from a theorem of Steiner*:

If the three perpendiculars from the vertices of one triangle on the sides of another are concurrent, the three corresponding perpendiculars from the vertices of the latter on the sides of the former are also concurrent.

The following proof is due to Mr W. J. C. Miller†:

Because
$$\angle E_2I_2A = \tfrac{1}{2}CAB = \angle F_3I_3A$$
$$\angle F_3I_3B = \tfrac{1}{2}ABC = \angle D_1I_1B$$
$$\angle D_1I_1C = \tfrac{1}{2}BCA = \angle E_2I_2C;$$

therefore I_1D_1, I_2E_2, I_3F_3 will meet in a point O_0 such that

$$O_0I_1 = O_0I_2 = O_0I_3;$$

hence O_0 is the circumcentre of $I_1I_2I_3$.

Similarly for the other triads, which meet at the points

$$O_1,\ O_2,\ O_3.$$

DEF. Mr Lemoine has proposed‡ to call triangles such as those of Steiner's theorem *orthologous*, and the points of concurrency of the perpendiculars *centres of orthology.*

Hence ABC is orthologous with each of the triangles

$$I_1I_2I_3,\quad II_3I_2,\quad I_3II_1,\quad I_2I_1I$$

and the respective centres of orthology are

$$I,\ O_0;\quad I_1,\ O_1;\quad I_2,\ O_2;\quad I_3,\ O_3.$$

(2) *The figures* $O_0I_2O_1I_3$, $O_0I_3O_2I_1$, $O_0I_1O_3I_2$ *are rhombi.*

For O_0I_2, O_1I_3 are perpendicular to AC

O_0I_3, O_1I_2 „ „ „ AB;

and
$$O_0I_2 = O_0I_3.$$

Hence
$$O_0O_1,\quad O_0O_2,\quad O_0O_3$$

are respectively perpendicular to

$$I_2I_3,\quad I_3I_1,\quad I_1I_2.$$

---

* Crelle's *Journal*, II. 287 (1827), or Steiner's *Gesammelte Werke*, I. 157 (1881).

† *Lady's and Gentleman's Diary* for 1863, p. 54.

‡ *Journal de Mathématiques Spéciales*, 3rd series, III. 63 (1889), and the memoir *Sur les triangles orthologiques* read at the Limoges meeting (1890) of the *Association Française pour l'avancement des Sciences.*

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(3) O_2O_3, O_3O_1, O_1O_2

are respectively parallel to

I_2I_3, I_3I_1, I_1I_2.

For O_2I_3, O_3I_2 are perpendicular to BC;
and they are equal, since the radii of the circumcircles of the four I triangles are equal;
therefore $O_2O_3I_2I_3$ is a parallelogram.

(4) The four O triangles are congruent to their respective four I triangles, and their corresponding sides are parallel.

(5) The points O_0, O_1, O_2, O_3 are the orthocentres of the four O triangles taken in order.

(6) *The figures* $IO_2I_1O_3$, $IO_3I_2O_1$, $IO_1I_3O_2$ *are rhombi.*

For IO_2, I_1O_3 are perpendicular to AC
IO_3, I_1O_2 ,, ,, ,, AB;
and $IO_2 = IO_3$,
since the radii of the circumcircles of the four I triangles are equal.

(7) The points I, I_1, I_2, I_3 are the circumcentres of the four O triangles taken in order.

(8) IO_0, I_1O_1, I_2O_2, I_3O_3
are the Euler's lines of the four I triangles, and of the four O triangles, and the circumcentre O of ABC is the mid point of each of them.

(9) By referring to the Section* on the nine-point circle, it will be seen that the circumcircle of ABC is the nine-point circle of the eight I and O triangles, and that the radii of the circumcircles of these eight triangles are each 2R.

It will also be seen that the circumcircle of ABC bisects each of the six straight lines

II_1, II_2, II_3, I_2I_3, I_3I_1, I_1I_2

* *Proceedings of the Edinburgh Mathematical Society*, XI. 19-57 (1893).

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at U , V , W , U' , V' , W' ;
and that UU' , VV' , WW'
are the diameters of the circumcircle ABC perpendicular to
$BC , CA , AB.$

(10) $U'V'W' , U'VW , UV'W , UVW'$
are the complementary triangles of the four I triangles taken in order.

(11) $O_0O_1 , O_0O_2 , O_0O_3 , O_2O_3 , O_3O_1 , O_1O_2$
pass respectively through the points
$U' , V' , W' , U , V , W .$

(12) The following pairs of straight lines intersect on the circumcircle of ABC:
$O_0O_1 , O_2O_3 ; O_0O_2 , O_3O_1 ; O_0O_3 , O_1O_2$
at $U_1 , V_1 , W_1 .$

(13) Triangle $U_1V_1W_1$ bears to $O_0O_1O_2$ exactly the same relations that ABC does to $I_1I_2I_3$.

FIGURE 64.

(14) Of the four I triangles taken in order let
G_0 , G_1 , G_2 , G_3
be the centroids; then the concurrency of

I_1U' , I_2V' , I_3W'	determines	G_0
IU' , I_3V , I_2W	,,	G_1
I_3U , IV' , I_1W	,,	G_2
I_2U , I_1V , IW'	,,	G_3.

(15)

G_0	lies on	IO_0	and	$IG_0 = 2O_0G_0$
G_1	,, ,,	I_1O_1	,,	$I_1G_1 = 2O_1G_1$
G_2	,, ,,	I_2O_2	,,	$I_2G_2 = 2O_2G_2$
G_3	,, ,,	I_3O_3	,,	$I_3G_3 = 2O_3G_3$.

(16) Through O pass *
$IG_0 , I_1G_1 , I_2G_2 , I_3G_3$ and
$OI = 3OG_0 , OI_1 = 3OG_1 , OI_2 = 3OG_2 , OI_3 = 3OG_3 .$

* Thomas Weddle in the *Lady's and Gentleman's Diary* for 1849, p. 76.

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(17) If through G_0 parallels be drawn to

$$O_0U', \quad O_0V', \quad O_0W'$$

these parallels will meet

$$\begin{array}{ll} & I\,U', \quad I\,V', \quad I\,W' \\ \text{at} & G_1\ , \quad G_2\ , \quad G_3\ . \end{array}$$

(18) $G_1G_2G_3$ is directly similar to $U'V'W'$ and $I_1I_2I_3$, the ratio of similitude in the first case being $2:3$, and in the second $1:3$.

(19) I, U, V, W; O_0, U', V', W'; G_0, G_1, G_2, G_3 form orthic tetrastigms.

FIGURE 63.

(20) The areas of the six rhombi *

$$\begin{array}{ll} & O_0I_2O_1I_3\ , \quad O_0I_3O_2I_1\ , \quad O_0I_1O_3I_2 \\ & I\,O_2I_1O_3\ , \quad I\,O_3I_2O_1\ , \quad I\,O_1I_3O_2 \\ \text{are} & 2Ra\ , \quad 2Rb\ , \quad 2Rc\ . \end{array}$$

(21) The areas of the three parallelograms †

$$\begin{array}{ll} & I_2I_3O_2O_3\ , \quad I_3I_1O_3O_1\ , \quad I_1I_2O_1O_2 \\ \text{are} & 2R(b+c), \quad 2R(c+a), \quad 2R(a+b)\ . \end{array}$$

(22) The figure $I_1O_3I_2O_1I_3O_2$ is an equilateral hexagon ‡; its opposite sides are parallel ‡, and equal to the diameter of the circumcircle ‡ of ABC; its angles are the supplements§ of the angles of ABC; and its area § is equal to the sum of the areas of $I_1I_2I_3$ and $O_1O_2O_3$, that is equal to $4Rs$.

---

* The last three are given by Rev. William Mason of Normanton in the *Lady's and Gentleman's Diary* for 1863, p. 53.

† Mr S. Constable in the *Educational Times*, XXXI. 113 (1878).

‡ T. S. Davies in the *Philosophical Magazine*, II. 32 (1827).

§ Rev. William Mason in the *Lady's and Gentleman's Diary* for 1863, p. 54.

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§ 7. Relations among Radii.

*The sum of the radii of the three excircles diminished by the radius of the incircle is double the diameter of the circumcircle.**

Figure 64.

Let O be the circumcentre, ID, I_1D_1, I_2D_2, I_3D_3 the radii of the incircle and the three excircles perpendicular to BC.

From O draw OA′ perpendicular to BC, and let OA′ meet the circumcircle below BC at U and above it at U′.

Because OA′ is perpendicular to BC,

therefore A′ is the mid point of BC, and U the mid point of arc BUC.

But since AI_1 bisects $\angle$ BAC,

therefore it bisects arc BUC, that is, AI_1 passes through U.

Since UU′ is a diameter, and $\angle UAI_3$ is right,

therefore I_2I_3 passes through U′.

Now because ID, UA′, I_1D_1 are parallel, and $DA' = D_1A'$,

therefore $$2UA' = I_1D_1 - ID\ ;$$

and because I_2D_2, U′A′, I_3D_3 are parallel, and $D_2A' = D_3A'$,

therefore $$2U'A' = I_2D_2 + I_3D_3\ .$$

Hence $$2(UA' + U'A') = I_1D_1 + I_2D_2 + I_3D_3 - ID\ ;$$

that is $$4R = r_1 + r_2 + r_3 - r\ .$$

(1) *The sum of the distances of the circumcentre from the sides of a triangle is equal to the sum of the radii of the incircle and the circumcircle; and the sum of the distances of the orthocentre from the vertices is equal to the sum of the diameters of the incircle and the circumcircle.*

* Feuerbach in his *Eigenschaften...des...Dreiecks*, §5 (1822), proves it algebraically. The proof in the text is that of T. S. Davies in the *Ladies' Diary* for 1835, p. 55.

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For $$OA' = OU - A'U$$
$$= R - \tfrac{1}{2}(r_1 - r).$$

Similarly $$OB' = R - \tfrac{1}{2}(r_2 - r);$$
$$OC' = R - \tfrac{1}{2}(r_3 - r);$$

therefore $$OA' + OB' + OC' = 3R - \tfrac{1}{2}(r_1 + r_2 + r_3 - 3r)$$
$$= 3R - \tfrac{1}{2}(4R + r - 3r)$$
$$= 3R - (2R - r)$$
$$= R + r.$$

Again $$HA + HB + HC = 2OA' + 2OB' + 2OC',$$
$$= 2(R + r).$$

If one of the angles of the triangle be obtuse, the circumcentre will fall outside the triangle, and its distance from the side opposite the obtuse angle must then be considered negative. Also if the circumcentre fall outside the triangle, so will the orthocentre. In that case the distance of the orthocentre from the vertex of the obtuse angle must be considered negative.

These two properties, as well as the remarks at the end of the proof, are given by Carnot in his *Géométrie de Position*, § 137 (1803).

The following is Carnot's mode of proving the first property.

FIGURE 65.

The quadrilaterals AB'OC', BA'OC', CB'OA' are inscriptible in circles; therefore

$$AO \cdot B'C' = AB' \cdot C'O + AC' \cdot B'O$$
$$BO \cdot C'A' = BC' \cdot A'O + BA' \cdot C'O$$
$$CO \cdot A'B' = CA' \cdot B'O + CB' \cdot A'O.$$

Adding these equations, and noting that $AO = BO = CO = R$, that $B'C' + C'A' + A'B' = s$, and that $AB' = CB'$, $AC' = BC'$, $BA = CA'$, we have

$$sR = s(A'O + B'O + C'O) - \tfrac{1}{2}(A'O \cdot BC + B'O \cdot CA + C'O \cdot AB)$$
$$= s(A'O + B'O + C'O) - \Delta$$
$$= s(A'O + B'O + C'O) - sr;$$

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therefore $$R = A'O + B'O + C'O - r,$$

or $$R + r = A'O + B'O + C'O.$$

(2) *The relation* $4R = r_1 + r_2 + r_3 - r$ *has been employed to establish* $R + r = OA' + OB' + OC'$;

but the method of procedure may be reversed.

FIGURE 64.

For $$OA' = OU - A'U = R - \tfrac{1}{2}(r_1 - r).$$

Similarly $$OB' = R - \tfrac{1}{2}(r_2 - r)$$

$$OC' = R - \tfrac{1}{2}(r_3 - r);$$

therefore $$OA' + OB' + OC' = 3R - \tfrac{1}{2}(r_1 + r_2 + r_3 - 3r)$$

that is $$R + r = 3R - \tfrac{1}{2}(r_1 + r_2 + r_3 - 3r);$$

whence $$4R = r_1 + r_2 + r_3 - r.$$

(3) $$A'U + B'V + C'W = 2R - r$$

and * $$A'U' + B'V' + C'W' = 4R + r.$$

These results follow from subtracting $A'O + B'O + C'O$ from $3R$, and adding $A'O + B'O + C'O$ to $3R$.

For another proof of the first of them see *Mathematical Questions from the Educational Times*, XVII. 47 (1872).

(4) *The following relations subsist between the distances of the circumcentre from the sides of a triangle and the radii of the circumcircle and the excircles*: †

$$-OA' + OB' + OC' = -R + r_1$$

$$OA' - OB' + OC' = -R + r_2$$

$$OA' + OB' - OC' = -R + r_3.$$

FIGURE 66.

From U draw UT perpendicular to AB,

and from O draw OR perpendicular to UT.

---

* Hind's *Trigonometry*, 4th ed., p. 309 (1841).

† Mr Bernh. Möllmann in Grunert's *Archiv*, XVII. 379 (1851).

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It may be proved that $AT = \frac{1}{2}(AB + AC)$

and $BT = \frac{1}{2}(AB - AC)$;

therefore $AB' = \frac{1}{2}AC = \frac{1}{2}(AT - BT)$

$= C'T = OR.$

Hence the right-angled triangles AB'O, ORU are congruent

and $OB' = UR;$

therefore $OB' + OC' = UT$

$= \frac{1}{2}(IF + I_1F_1);$

therefore $2(OB' + OC') = r + r_1.$

Now $OA' + OB' + OC' = R + r;$

therefore $-OA' + OB' + OC' = -R + r_1.$

(5) *The following is another proof* of the relation*

$$OA' + OB' + OC' = R + r.$$

FIGURE 67.

Through I the incentre draw a parallel to AC meeting OB' in K; through K draw a parallel to IC meeting OA' in L and BC in N. From N draw NM perpendicular to AC and meeting OA' in M.

Since KN is parallel to the bisector of $\angle ACB$,

therefore $$CN = IK = EB' = CB' - CE$$

$$= \frac{AC}{2} - \frac{BC + CA - AB}{2}$$

$$= \frac{AB - BC}{2};$$

therefore $$A'N = A'C + CN = \frac{AB}{2} = C'B.$$

Again, since MN is perpendicular to AC,

therefore $\angle MNA' = 90° - \angle ACB$

$= 90° - \angle C'OB = \angle OBC';$

* Mr Lemoine in *Journal de Mathématiques Élémentaires*, 2nd series, IV. 217-8 (1885). (6) also is his.

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therefore the right-angled triangles MNA′, OBC′ are congruent,

and $$A'M = OC', \quad MN = OB = R.$$

Since
$$\begin{aligned} \angle MNL &= \angle MNA' + \angle A'NL \\ &= 90° - C + \frac{C}{2} \\ &= 90° - \frac{C}{2} \\ &= \angle MLN ; \end{aligned}$$

therefore triangle MLN is isosceles;

therefore „ OLK „ „ .

Hence
$$\begin{aligned} OA' + OB' + OC' &= OA' + OK + KB' + A'M \\ &= OA' + OL + IE + A'M \\ &= LM + IE \\ &= MN + IE \\ &= R + r . \end{aligned}$$

(6) If r_1 denote the radius of the first excircle, it may be shown by an analogous proof that

$$OB' + OC' - OA' = r_1 - R.$$

Hence the theorem:

If in a triangle the radius of an excircle be equal to the radius of the circumcircle, one of the three distances of the orthocentre from the vertices is equal to the sum of the other two, and conversely.

(7) *If* D, D' *be the projections of* A' *on* OB, OC,
E, E' „ „ „ „ B' „ OC, OA,
F, F' „ „ „ „ C' „ OA, OB

*then** $$\sqrt{\frac{DD'}{a}} + \sqrt{\frac{EE'}{b}} + \sqrt{\frac{FF'}{c}} = \frac{R+r}{R}.$$

FIGURE 68.

Since triangle OBC is isosceles, DD′ is parallel to BC,

and $$\frac{DD'}{a} = \frac{OD}{R}.$$

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* Mr J. Somméritis of Chalcis in Vuibert's *Journal de Mathématiques Élémentaires*, XVI. 128 (1892). The solution in the text is that given on p. 141.

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From the right-angled triangles OA'B, ODA'

$$OD = \frac{OA'^2}{R};$$

therefore $$\frac{DD'}{a} = \frac{OA'^2}{R^2} \text{ and } \sqrt{\frac{DD'}{a}} = \frac{OA'}{R}.$$

Similarly $$\sqrt{\frac{EE'}{b}} = \frac{OB'}{R} \text{ and } \sqrt{\frac{FF'}{c}} = \frac{OC'}{R};$$

therefore $$\sqrt{\frac{DD'}{a}} + \sqrt{\frac{EE'}{b}} + \sqrt{\frac{FF'}{c}} = \frac{OA' + OB' + OC'}{R}$$

$$= \frac{R + r}{R}.$$

(8) *The potency (or power) of the incentre of a triangle with respect to the circumcircle is equal to twice the rectangle under the radii of the incircle and the circumcircle.**

FIGURE 69.

In ABC let O be the circumcentre, I the incentre.

Join OI.

Through O draw U'U the diameter of the circumcircle perpendicular to BC. Then U is the mid point of the arc BUC, and AU will pass through I.

Join CU, CU', CI, AO, and from I draw IE the radius of the incircle perpendicular to AC.

Then $$\angle UIC = \angle IAC + \angle ICA,$$

$$= \tfrac{1}{2}(A + C);$$

and $$\angle UCI = \angle BCI + \angle BCU,$$

$$= \angle BCI + \angle BAU,$$

$$= \tfrac{1}{2}(A + C);$$

therefore $$CU = IU.$$

---

* William Chapple in *Miscellanea Curiosa Mathematica*, I. 123 (1746). Euler gave the property in an inconvenient form about twenty years later. A tolerably full history of Chapple's theorem and its developments during the 18th century will be found in the *Proceedings of the Edinburgh Mathematical Society*, V. 62-78 (1887).

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Again, the right-angled triangles AEI, U'CU are similar;

therefore $AI : IE = U'U : UC$;

therefore $AI \cdot UC = U'U \cdot IE$,

that is $AI \cdot IU = 2Rr$.

Lastly from the isosceles triangle OAU

$$OA^2 - OI^2 = AI \cdot IU,$$

that is $$R^2 - OI^2 = 2Rr.$$

(9) If OI be denoted by d, then

$$R^2 - d^2 = 2Rr,$$

or $$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}.$$

(10) *The potency (or power) of an excentre of a triangle with respect to the circumcircle is equal to twice the rectangle under the radii of the excircle and the circumcircle.**

FIGURE 70.

In ABC let O be the circumcentre, I_1 an excentre.

Join OI_1.

Through O draw U'U the diameter of the circumcircle perpendicular to BC. Then U is the mid point of the arc BUC, and AU will pass through I_1,

Join CU, CU', CI_1, AO, and from I_1 draw I_1E_1 the radius of the excircle perpendicular to AC.

Then $$\angle U I_1C = 180° - (\angle I_1AC + \angle I_1CA),$$
$$= 90° - \tfrac{1}{2}(A+C);$$

and $$\angle U C I_1 = \angle BCI_1 - \angle BCU,$$
$$= \angle BCI_1 - \angle BAU,$$
$$= 90° - \tfrac{1}{2}(A+C);$$

therefore $$CU = I_1U.$$

Again, the right-angled triangles AE_1I_1, U'CU are similar;

therefore $AI_1 : I_1E_1 = U'U : UC$;

therefore $AI_1 \cdot UC = U'U \cdot I_1E_1$,

that is $AI_1 \cdot I_1U = 2Rr_1$.

* John Landen in *Lucubrationes Mathematicae*, pp. 1-6 (1755).

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Lastly from the isosceles triangle OAU,

$$OI_1^2 - OA^2 = AI_1 \cdot I_1U,$$

that is $OI_1^2 - R^2 = 2Rr_1.$

Hence also $OI_2^2 - R^2 = 2Rr_2,$

and $OI_3^2 - R^2 = 2Rr_3.$

(11) If OI_1, OI_2, OI_3 be denoted by d_1, d_2, d_3 then

$$d_1^2 - R^2 = 2Rr_1,\ d_2^2 - R^2 = 2Rr_2,\ d_3^2 - R^2 = 2Rr_3;$$

or

$$\frac{1}{R+d_1} + \frac{1}{R-d_1} = -\frac{1}{r_1}$$

$$\frac{1}{R+d_2} + \frac{1}{R-d_2} = -\frac{1}{r_2}$$

$$\frac{1}{R+d_3} + \frac{1}{R-d_3} = -\frac{1}{r_3}.$$

(12) The potency of I with respect to the circumcircle is *

$$\frac{abc}{a+b+c}.$$

For $$2Rr = \frac{abc}{2\Delta} \cdot \frac{2\Delta}{a+b+c}.$$

(13) The potency of I_1 with respect to the circumcircle is

$$\frac{abc}{-a+b+c}.$$

For $$2Rr_1 = \frac{abc}{2\Delta} \cdot \frac{2\Delta}{-a+b+c}.$$

(14) If the first excircle cut the circumcircle at S, and I_1S be produced to intersect the circumcircle at T, then $I_1T = 2R$.

For $I_1S \cdot I_1T$ = potency of I_1 with respect to circumcircle,

$= OI_1^2 - R^2 = 2Rr_1;$

and $I_1S = r_1.$

* C. J. Matthes, *Commentatio de Proprietatibus Quinque Circulorum*, p. 41 (1831).

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(15) If IO be produced to meet the circumcircle in M, N, and the incircle in P, Q (the order of the letters is MPOIQN), then *

$$MP \cdot NQ = r^2$$
$$MQ \cdot NP = 4Rr + r^2.$$

For $MP = (R - r) + OI$, $NQ = (R - r) - OI$
and $MQ = (R + r) + OI$, $NP = (R + r) - OI$.

(16) If I_1O be produced to meet the circumcircle in M, N, and the first excircle in P, Q (the order of the letters is $MOQNI_1P$), then

$$MP \cdot NQ = r_1^2$$
$$MQ \cdot NP = 4Rr_1 - r_1^2.$$

(17)
$$IM \cdot IN = 2Rr$$
$$OP \cdot OQ = -R^2 + 2Rr + r^2.$$

(18)
$$I_1M \cdot I_1N = 2Rr_1$$
$$OP \cdot OQ = R^2 + 2Rr_1 - r_1^2.$$

(19) The product of the potencies † of P and Q with respect to the circumcircle

$$MP \cdot NP \times MQ \cdot NQ = r^3(4R + r).$$

The product of the potencies of M and N with respect to the incircle

$$MP \cdot MQ \times NP \cdot NQ = r^3(4R + r).$$

(20) The product of the potencies of P and Q with respect to the circumcircle

$$MP \cdot NP \times MQ \cdot NQ = r_1^3(4R - r_1).$$

The product of the potencies of M and N with respect to the first excircle

$$MP \cdot MQ \times NP \cdot NQ = r_1^3(4R - r_1).$$

---

* The first part is given by Mr Neorouzian in the *Nouvelles Annales*, IX. 216-7 (1850); the second part occurs in *Exercices de Géométrie*, by F.I.C., 2nd ed., p. 506 (1882).

† The first part is given in *Nouvelles Annales*, XVII. 358, 447-8 (1858), and attributed to Grunert.

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(21) The radius of the circumcircle is never less than the diameter of the incircle.*

For OI^2 is positive;

therefore $R - 2r$ cannot be negative.

(22) When the radius of the circumcircle is equal to the diameter of the incircle, the circumcentre and the incentre coincide, and the triangle is equilateral.

(23) When the straight line joining the incentre and the circumcentre passes through one of the vertices, the triangle is isosceles.

(24) Since the value of OI^2 is independent of the sides of the triangle ABC, if two circles whose radii are R and r be so situated that the square of the distance between their centres equals $R(R - 2r)$, then any number of triangles may be drawn, each of which shall be inscribed in the larger circle, and circumscribed about the smaller†; and if the two circles be so situated that the square of the distance between their centres is not equal to $R(R - 2r)$, then no triangle can be inscribed and circumscribed.

(25) Since the value of OI_1^2 is independent of the sides of the triangle, a corresponding statement may be made regarding two circles whose radii are R and r_1.

(26) If one side of a triangle inscribed in and circumscribed about two given circles be given, the other two sides may be found.

(27) Of the innumerable triangles that may be inscribed in and circumscribed about two given circles, two will be isosceles; and the common diameter of the two circles will pass through their vertices and cut their bases at right angles. That isosceles triangle which has the least base and the greatest altitude will be the greatest, and the other isosceles triangle will be the least of all the triangles that can be inscribed and circumscribed.

---

* Theorems (21)–(24), (26), (27) are given by Chapple; (28) part of which is given by Chapple, is due to Dr Otto Böklen. See Grunert's *Archiv*, XXXVIII. 143 (1862).

† A detailed proof of this statement, if such should be considered necessary, is given by Dr W. H. Besant in the *Quarterly Journal of Mathematics*, XII. 276 (1873).

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(28) In connection with these innumerable triangles a large number of constant magnitudes may be found. A few are here enumerated.

(*a*) The sum of the perpendiculars from the circumcentre to the sides is constant.

(*b*) The sum of the distances of the orthocentre from the vertices is constant.

(*c*) The sum of the radii of the excircles is constant.

(*d*) The sum of the reciprocals of the radii of the excircles is constant.

(*e*) The ratio of the product of the sides to the sum of the sides is constant.

(*f*) The ratio of the area to the perimeter is constant.

The proofs of these statements are

(*a*) $$OA' + OB' + OC' = R + r$$

(*b*) $$\tfrac{1}{2}(HA + HB + HC) = R + r$$

(*c*) $$r_1 + r_2 + r_3 = R + r$$

(*d*) $$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$

(*e*) $$\frac{abc}{a+b+c} = 2Rr$$

(*f*) $$\frac{\Delta}{s} = r \quad .$$

(29) The sum of the squares of the distances of the circumcentre from the incentre and the excentres is equal to three times the square of the diameter of the circumcircle.*

For $$\Sigma(OI^2) = 4R^2 + 2R(r_1 + r_2 + r_3 - r)$$
$$= 4R^2 + 2R \cdot 4R$$
$$= 12R^2.$$

* Feuerbach, *Eigenschaften...des...Dreiecks*, § 50 (1822).

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(30) The preceding theorem may be derived from the following*:

The sum of the squares of the tangents drawn from the centres of the four circles of contact of a triangle to any circle which passes through the circumcentre is equal to three times the square of the diameter of the circumcircle.

FIGURE 71.

Let Q be the centre of a circle passing through O the circumcentre.

Draw the diameter of the circumcircle UU′ perpendicular to BC and bisecting II_1 at U and I_2I_3 at U′.

Join Q with O, I, I_1, I_2, I_3, U, U′, and draw CU, CU′.

If the four tangents be denoted by t, t_1, t_2, t_3,

$$\begin{aligned}
\text{then } t^2+t_1^2+t_2^2+t_3^2 &= (Q\,I^2 - QO^2) + (Q\,I_1^2 - Q\,O^2) \\
&+ (Q\,I_2^2 - QO^2) + (Q\,I_3^2 - Q\,O^2) \\
&= (Q\,I^2 + Q\,I_1^2) + (Q\,I_2^2 + Q\,I_3^2) - 4QO^2 \\
&= 2(QU^2 + U\,I^2) + 2(QU'^2 + U'I_2^2) - 4QO^2 \\
&= 2(QU^2 + UC^2) + 2(QU'^2 + U'C^2) - 4QO^2 \\
&= 2(QU^2 + QU'^2) + 2(UC^2 + U'C^2) - 4QO^2 \\
&= 4(QO^2 + OU^2) + 2\,U'U^2 - 4QO^2 \\
&= 4\,OU^2 + 8OU^2 \\
&= 12R^2.
\end{aligned}$$

When QO becomes zero, or the circle with centre Q vanishes to a point,

$$t^2+t_1^2+t_2^2+t_3^2 = OI^2+OI_1^2+OI_2^2+OI_3^2.$$

(31) Since $-\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 0,$

therefore $-\frac{1}{2Rr} + \frac{1}{2Rr_1} + \frac{1}{2Rr_2} + \frac{1}{2Rr_3} = 0;$

therefore $\frac{1}{d^2-R^2} + \frac{1}{d_1^2-R^2} + \frac{1}{d_2^2-R^2} + \frac{1}{d_3^2-R^2} = 0;$

therefore $\frac{1}{12(R^2-d^2)} + \frac{1}{12(R^2-d_1^2)} + \frac{1}{12(R^2-d_2^2)} + \frac{1}{12(R^2-d_3^2)} = 0.$

* Philip Beecroft in the *Lady's and Gentleman's Diary* for 1845, p. 63.

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[illegible]

[illegible]

[illegible]

§ 8. Area.

The area of a triangle is a mean proportional between the rectangle contained by the semiperimeter and its excess above any one side and the rectangle contained by its excesses above the other two sides.

First Demonstration.

Figure 72.

Let ABC be the given triangle, and let each of its sides be given: to find the area.

Inscribe in the triangle the circle DEF whose centre is I, and join I with the points A, B, C, D, E, F.

Then the rectangle $BC \cdot ID$ = twice triangle BCI,

the rectangle $CA \cdot IE$ = twice triangle CAI,

and the rectangle $AB \cdot IF$ = twice triangle ABI;

hence the rectangle under the perimeter of triangle ABC and ID, the radius of the circle DFE = twice triangle ABC.

Produce BC, and make CD_2 equal to AE; then BD_2 is the semiperimeter, and the rectangle $BD_2 \cdot ID$ = triangle ABC.
But the rectangle $BD_2 \cdot ID$ is a side of the solid contained by BD_2 and the square of ID; therefore the area of the triangle will be a side of the solid contained by BD_2 and the square of ID.

Draw IL perpendicular to IB, CL perpendicular to CB, and join BL.

Since each of the angles BIL, BCL is right,
the points B, I, C, L are concyclic;
therefore the angles BIC, BLC are equal to two right angles.
But the angles BIC, AIE are equal to two right angles,
because AI, BI, CI bisect the angles at the point I;
therefore angle AIE = angle BLC,
and triangle AIE is similar to triangle BLC.

Hence $BC : LC = AE : IE$,

$= CD_2 : ID$;

therefore $BC : CD_2 = LC : ID$, by alternation,

$= CK : DK$;

and $BD_2 : CD_2 = CD : DK$, by composition.

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Consequently $BD_2^2 : BD_2 \cdot CD_2 = CD \cdot BD : BD \cdot DK$,

$= CD \cdot BD : ID^2$;

therefore $BD_2^2 \cdot ID^2 = BD_2 \cdot CD_2 \cdot BD \cdot CD$.

Now each of the lines BD_2, CD_2, BD, CD is given ;

for BD_2 is the semiperimeter, CD_2 the excess of the semiperimeter above BC, BD the excess of the semiperimeter above AC, and CD the excess of the semiperimeter above AB.

The area of the triangle therefore is given.

[Numerical illustration.]

Let AB consist of 13 parts, BC of 14, CA of 15.

Add the three together ; the result is 42, of which the half is 21. Subtract 13 ; there remain 8: 14, there remain 7 : 15, there remain 6. 21, 8, 7, 6 into one another produce 7056, the square root of which is 84.

The area of the triangle is 84.

This useful theorem occurs in a treatise "On the Dioptra" (*περὶ διόπτρας*) which many mathematical historians attribute to Heron of Alexandria (about 120 B.C.). See Cantor's *Vorlesungen über Geschichte der Mathematik*, I. 322-6 (1880). Mr Maximilien Marie, however (*Histoire des Sciences Mathématiques et Physiques*, I. 177-190), thinks the theorem cannot belong to so early a period, and ascribes it to Heron of Constantinople. The theorem was known to the Hindu mathematician Brahmegupta (born 598 A.D.) and to the Arabs. A good deal of historical information regarding it will be found in Chasles' *Aperçu Historique*, Note XII.

I have translated the demonstration in the text from Hultsch's *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, pp. 235-7 (1864), but I have not transliterated the notation.

Second Demonstration.

Figure 36.

Let ABC be a triangle, AX the perpendicular* from A to BC.

Then $AB^2 = BC^2 + CA^2 - 2BC \cdot CX$

that is $c^2 = a^2 + b^2 - 2a \cdot CX$;

therefore $CX = \dfrac{a^2 + b^2 - c^2}{2a}$.

* Whatever be the shape of the triangle one of the perpendiculars will always fall inside the triangle. Let that perpendicular be AX.

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Now $$AX^2 = AC^2 - CX^2$$

$$= b^2 - \left(\frac{a^2+b^2-c^2}{2a}\right)^2$$

$$= \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4a^2}$$

$$= \frac{2s \cdot 2s_1 \cdot 2s_2 \cdot 2s_3}{4a^2};$$

therefore $$AX = \frac{2}{a}\sqrt{ss_1s_2s_3}.$$

Hence $$\triangle = \tfrac{1}{2}BC \cdot AX$$

$$= \frac{a}{2} \cdot \frac{2}{a}\sqrt{ss_1s_2s_3}$$

$$= \sqrt{ss_1s_2s_3}.$$

THIRD DEMONSTRATION.

FIGURE 28.

Because triangles AFI, AF_1I_1 are similar,

therefore $$AF : IF = AF_1 : I_1F_1;$$

therefore $$AF_1 \cdot AF : AF_1 \cdot IF = AF_1 \cdot IF : I_1F_1 \cdot IF.$$

Because triangles IBF, BI_1F_1 are similar,

therefore $$BF : IF = I_1F_1 : BF_1;$$

therefore $$IF \cdot I_1F_1 = BF \cdot BF_1.$$

Hence $$AF_1 \cdot AF : AF_1 \cdot IF = AF_1 \cdot IF : BF \cdot BF_1;$$

therefore $$s\,s_1 : \triangle = \triangle : s_2s_3.$$

(1) $$\triangle = \tfrac{1}{4}\sqrt{2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4)}.$$

This expression is convenient when a, b, c are irrational quantities.

(2) *The following method will enable us to discover the expression for the area of a triangle, if it is known that the square of its area is an integral function of its sides.**

* Terquem in *Nouvelles Annales*, III. 219-220 (1844). The method is also applied by Terquem to find the expression for the area of a cyclic quadrilateral, and it had previously been applied by P. L. Cirodde in *Nouvelles Annales*, I. 117 (1842), to find the volume of a spherical segment when it is known that the volume is a function of the third degree of its height.

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Let $\triangle$ denote the area of the triangle, a, b, c its sides.

Then $\triangle^2$ is a symmetrical function of the sides of the fourth degree. If one side becomes equal to the sum of the two others, the area vanishes;

therefore $\triangle^2$ contains the three factors $-a+b+c$, $a-b+c$, $a+b-c$. The fourth factor must therefore be of the form $m(a+b+c)$, where m is a constant number;

therefore $\triangle^2 = m(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.

To determine the value of m, suppose the three sides of the triangle to be equal;

then $$\triangle^2 = 3ma^4.$$

But the square of the area of an equilateral triangle,

whose side is a, $$= \frac{3}{16}a^4;$$

therefore $$m = \frac{1}{16}.$$

Formulae for the areas of certain triangles.

See the notation, pp. 7–11.

Triangles connected with the Centroid.

$$A'B'C' = AC'B' = C'BA' = B'A'C = \tfrac{1}{4}\triangle \tag{1}$$

If R, S, T be the projections of G on the sides

$$GST = \frac{4}{9}\frac{\triangle^3}{b^2c^2} \qquad GTR = \frac{4}{9}\frac{\triangle^3}{c^2a^2} \qquad GRS = \frac{4}{9}\frac{\triangle^3}{a^2b^2} \tag{2}$$

$$RST = \frac{4}{9}\cdot\frac{a^2+b^2+c^2}{a^2b^2c^2}\triangle^3 \tag{3}$$

Triangles connected with the Circumcentre.

$$\left.\begin{aligned} OBC &= \tfrac{1}{2}ak_1 = \frac{a^2(-a^2+b^2+c^2)}{16\triangle} \\ OCA &= \tfrac{1}{2}bk_2 = \frac{b^2(\ a^2-b^2+c^2)}{16\triangle} \\ OAB &= \tfrac{1}{2}ck_3 = \frac{c^2(\ a^2+b^2-c^2)}{16\triangle} \end{aligned}\right\} \tag{4}$$

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Since O is the orthocentre of A'B'C',

therefore OC'B', C'OA', B'A'O

stand in the same relation to A'B'C' as

HCB, CHA, BAH

stand to ABC.

Hence expressions for the areas of these triangles may be derived from (27).

Triangles connected with the Incentre and Excentres.

$$\frac{DEF}{r} = \frac{D_1E_1F_1}{r_1} = \frac{D_2E_2F_2}{r_2} = \frac{D_3E_3F_3}{r_3} = \frac{\Delta}{2R} \tag{5}$$

$$D_1E_1F_1 + D_2E_2F_2 + D_3E_3F_3 - DEF = 2\Delta \tag{6}$$

$$DEF \cdot D_1E_1F_1 \cdot D_2E_2F_2 \cdot D_3E_3F_3 = \frac{\Delta^6}{16R^4} \tag{7}$$

$$\frac{1}{D_1E_1F_1} + \frac{1}{D_2E_2F_2} + \frac{1}{D_3E_3F_3} - \frac{1}{DEF} = 0 \tag{8}$$

$$\left.\begin{aligned} \frac{AEF \cdot BFD \cdot CDE}{IEF \cdot IFD \cdot IDE} &= \frac{\Delta^2}{r^4} \\ \frac{AE_1F_1 \cdot BF_1D_1 \cdot CD_1E_1}{I_1E_1F_1 \cdot I_1F_1D_1 \cdot I_1D_1E_1} &= \frac{\Delta^2}{r_1^4} \end{aligned}\right\} \tag{9}$$

and so on.

$$\frac{AEF \cdot BFD \cdot CDE}{(DEF)^2} = \frac{AE_1F_1 \cdot BF_1D_1 \cdot CD_1E_1}{(D_1E_1F_1)^2} = \cdots = \frac{\Delta}{4} \tag{10}$$

$$\Delta_0 = 2Rs, \quad \Delta_1 = 2Rs_1, \quad \Delta_2 = 2Rs_2, \quad \Delta_3 = 2Rs_3 \tag{11}$$

$$\Delta_0\Delta_1\Delta_2\Delta_3 = 16R^4\Delta^2 \tag{12}$$

$$\left.\begin{aligned} \Delta_0 : \Delta &= 2R : r \\ \Delta_1 : \Delta &= 2R : r \end{aligned}\right\} \tag{13}$$

and so on.

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$$\frac{\Delta}{\Delta_1}+\frac{\Delta}{\Delta_2}+\frac{\Delta}{\Delta_3}-\frac{\Delta}{\Delta_0}=2 \tag{14}$$

$$\left.\begin{aligned} 2\Delta\Delta_0 &= abcs = 4Rr_1r_2r_3 \\ 2\Delta\Delta_1 &= abcs_1 = 4Rr\,r_2r_3 \end{aligned}\right\} \tag{15}$$

and so on.

$$2r\Delta_0=2r_1\Delta_1=2r_2\Delta_2=2r_3\Delta_3=abc \tag{16}$$

$$\left.\begin{aligned} \Delta_0 : \Delta &= \Delta : DEF \\ \Delta_1 : \Delta &= \Delta : D_1E_1F_1 \end{aligned}\right\} \tag{17}$$

and so on.

$$\left.\begin{aligned} \Delta_0 : DEF &= 4R^2 : r^2 \\ \Delta_1 : D_1E_1F_1 &= 4R^2 : r_1^2 \end{aligned}\right\} \tag{18}$$

and so on.

$$\left.\begin{aligned} \Delta_0^2 &= R(r_2+r_3)(r_3+r_1)(r_1+r_2) \\ \Delta_1^2 &= R(r_2+r_3)(r_2-r)(r_3-r) \\ \Delta_2^2 &= R(r_3+r_1)(r_3-r)(r_1-r) \\ \Delta_3^2 &= R(r_1+r_2)(r_1-r)(r_2-r) \end{aligned}\right\} \tag{19}$$

Corresponding expressions may be obtained for

ABC, HCB, CHA, BAH

by substituting instead of

$$R,\quad r,\quad r_1,\quad r_2,\quad r_3$$
$$\tfrac{1}{2}R,\quad \rho,\quad \rho_1,\quad \rho_2,\quad \rho_3$$

$$\left.\begin{aligned} 4\Delta_0 &= 2(a_2+a_3)\,a_1 = 2(\beta_3+\beta_1)\,\beta_2 = 2(\gamma_1+\gamma_2)\,\gamma_3 \\ &= (a_1-a)(a_2+a_3)+(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_3-\gamma)(\gamma_1+\gamma_2) \\ 4\Delta_1 &= 2(a_2+a_3)\,a = 2(\beta_2-\beta)\,\beta_3 = 2(\gamma_3-\gamma)\,\gamma_2 \\ &= -(a_1-a)(a_2+a_3)+(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_2-\gamma)(\gamma_1+\gamma_2) \\ 4\Delta_2 &= 2(a_1-a)\,a_2 = 2(\beta_3+\beta_1)\,\beta = 2(\gamma_3-\gamma)\,\gamma_1 \\ &= (a_1-a)(a_2+a_3)-(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_3-\gamma)(\gamma_1+\gamma_2) \\ 4\Delta_3 &= 2(a_1-a)\,a_2 = 2(\beta_2-\beta)\,\beta_1 = 2(\gamma_1+\gamma_2)\,\gamma \\ &= (a_1-a)(a_2+a_3)+(\beta_2-\beta)(\beta_3+\beta_1)-(\gamma_3-\gamma)(\gamma_1+\gamma_2) \end{aligned}\right\} \tag{20}$$

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$$\left.\begin{aligned} \mathrm{I_1BC} &= \frac{ar_1}{2} = \frac{r_1^2(r_2+r_3)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}} \\ \mathrm{AI_2C} &= \frac{br_2}{2} = \frac{r_2^2(r_3+r_1)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}} \\ \mathrm{ABI_3} &= \frac{cr_3}{2} = \frac{r_3^2(r_1+r_2)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}} \end{aligned}\right\} \quad (21)$$

The three preceding triangles are similar to $\mathrm{I_1I_2I_3}$ and expressions for them may be derived from the expressions for Δ_0 by the comparison of homologous lines in the triangles.

Thus in $\mathrm{I_1BC}$, $\mathrm{I_1I_2I_3}$ the perpendiculars $\mathrm{I_1D_1}$, $\mathrm{I_1A}$, or r_1, a_1, are homologous lines;

therefore $$\mathrm{I_1BC} : \Delta_0 = r_1^2 : a_1^2.$$

Similarly expressions may be found for

triangles	which are similar to
$\mathrm{I\,BC}$, $\mathrm{AI_3C}$, $\mathrm{ABI_2}$	$\mathrm{I\,I_3I_2}$ or Δ_1
$\mathrm{I_3BC}$, $\mathrm{AI\,C}$, $\mathrm{ABI_1}$	$\mathrm{I_3I\,I_1}$ or Δ_2
$\mathrm{I_2BC}$, $\mathrm{AI_1C}$, ABI	$\mathrm{I_2I_1I}$ or Δ_3.

From these again, by making the appropriate changes in the letters, corresponding expressions may be found for

triangles	which are similar to
AYZ, XBZ, XYC	ABC
HYZ, XCZ, XYB	HCB
CYZ, XHZ, XYA	CHA
BYZ, XAZ, XYH	BAH.

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Triangles connected with the Angular Bisectors.

$$\left.\begin{aligned} \mathrm{L\,M\,N} &= \frac{2abc\triangle}{(b+c)(c+a)(a+b)} = \frac{l_1 l_2 l_3}{4s} \\ \mathrm{L\,M'N'} &= \frac{2abc\triangle}{(b+c)(a-c)(a-b)} = \frac{l_1 \lambda_2 \lambda_3}{4s_1} \\ \mathrm{L'M\,N'} &= \frac{2abc\triangle}{(b-c)(c+a)(a-b)} = \frac{\lambda_1 l_2 \lambda_3}{4s_2} \\ \mathrm{L'M'N} &= \frac{2abc\triangle}{(b-c)(a-c)(a+b)} = \frac{\lambda_1 \lambda_2 l_3}{4s_3} \end{aligned}\right\} \quad (22)$$

$$\left.\begin{aligned} \mathrm{L\,L'M} &= \frac{2abc\triangle}{(c+a)(b^2-c^2)} \quad \mathrm{L\,L'N} = \frac{2abc\triangle}{(a+b)(b^2-c^2)} \\ \mathrm{M\,M'N} &= \frac{2abc\triangle}{(a+b)(a^2-c^2)} \quad \mathrm{M\,M'L} = \frac{2abc\triangle}{(b+c)(a^2-c^2)} \\ \mathrm{N\,N'L} &= \frac{2abc\triangle}{(b+c)(a^2-b^2)} \quad \mathrm{N\,N'M} = \frac{2abc\triangle}{(c+a)(a^2-b^2)} \end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned} \mathrm{L\,L'M'} &= \frac{2abc\triangle}{(a-c)(b^2-c^2)} \quad \mathrm{L\,L'N'} = \frac{2abc\triangle}{(a-b)(b^2-c^2)} \\ \mathrm{M\,M'N'} &= \frac{2abc\triangle}{(a-b)(a^2-c^2)} \quad \mathrm{M\,M'L'} = \frac{2abc\triangle}{(b-c)(a^2-c^2)} \\ \mathrm{N\,N'L'} &= \frac{2abc\triangle}{(b-c)(a^2-b^2)} \quad \mathrm{N\,N'M'} = \frac{2abc\triangle}{(a-c)(a^2-b^2)} \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} \mathrm{L\,M\,N} &: \triangle_0 = 4s_1 s_2 s_3 : (b+c)(c+a)(a+b) \\ \mathrm{L\,M'N'} &: \triangle_1 = 4s\, s_2 s_3 : (b+c)(a-c)(a-b) \\ \mathrm{L'M\,N'} &: \triangle_2 = 4s\, s_3 s_1 : (b-c)(c+a)(a-b) \\ \mathrm{L'M'N} &: \triangle_3 = 4s\, s_1 s_2 : (b-c)(a-c)(a+b) \end{aligned}\right\} \quad (25)$$

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Triangles connected with the Orthocentre.

$$\left.\begin{aligned} ABX &= -\frac{(a^2-b^2+c^2)\Delta}{2a^2} & ACX &= \frac{(a^2+b^2-c^2)\Delta}{2a^2} \\ BCY &= \frac{(a^2+b^2-c^2)\Delta}{2b^2} & BAY &= \frac{(-a^2+b^2+c^2)\Delta}{2b^2} \\ CAZ &= \frac{(-a^2+b^2+c^2)\Delta}{2c^2} & CBZ &= \frac{(a^2-b^2+c^2)\Delta}{2c^2} \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned} \Delta_a = HCB &= \frac{(a^2-b^2+c^2)(a^2+b^2-c^2)}{16\Delta} \\ \Delta_b = CHA &= \frac{(a^2+b^2-c^2)(-a^2+b^2+c^2)}{16\Delta} \\ \Delta_c = BAH &= \frac{(-a^2+b^2+c^2)(a^2-b^2+c^2)}{16\Delta} \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} HBX &= \frac{(a^2-b^2+c^2)^2(a^2+b^2-c^2)}{32a^2\Delta} \\ HCX &= \frac{(a^2-b^2+c^2)(a^2+b^2-c^2)^2}{32a^2\Delta} \end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned} AYZ = \frac{(-a^2+b^2+c^2)^2\Delta}{4b^2c^2}, \quad XBZ &= \frac{(a^2-b^2+c^2)^2\Delta}{4c^2a^2}, \\ XYC &= \frac{(a^2+b^2-c^2)^2\Delta}{4a^2b^2} \end{aligned}\right\} \quad (29)$$

$$XYZ = \frac{(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)\Delta}{4a^2b^2c^2} \quad (30)$$

$$\left.\begin{aligned} DEF &: XYZ = r &&: 2\rho \\ D_1E_1F_1 &: XYZ = r_1 &&: 2\rho \end{aligned}\right\} \quad (31)$$

and so on.

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$$\left.\begin{aligned} \Delta_0 \;:\; \Delta &= 4s \;:\; x+y+z \\ \Delta_1 \;:\; \Delta &= 4s_1 : x+y+z \end{aligned}\right\} \qquad (32)$$

and so on.

Miscellaneous.

$$\left.\begin{aligned} \mathrm{I\,I_1O}\cdot 8\Delta &= abc\,(c-b) \\ \mathrm{I\,I_2O}\cdot 8\Delta &= abc\,(c-a) \\ \mathrm{I\,I_3O}\cdot 8\Delta &= abc\,(a-b) \end{aligned}\right\} \qquad (33)$$

$$\left.\begin{aligned} \mathrm{I\,I_1H}\cdot 8\Delta &= a(c-b)\,(b^2+c^2-a^2) \\ \mathrm{I\,I_2H}\cdot 8\Delta &= b(c-a)\,(c^2+a^2-b^2) \\ \mathrm{I\,I_3H}\cdot 8\Delta &= c(a-b)\,(a^2+b^2-c^2) \end{aligned}\right\} \qquad (34)$$

$$\left.\begin{aligned} (2\,\mathrm{I_2I_3I}-\mathrm{I_2I_3O})\,8\Delta &= abc\,(b+c) \\ (2\,\mathrm{I_3I_1I}-\mathrm{I_3I_1O})\,8\Delta &= abc\,(c+a) \\ (2\,\mathrm{I_1I_2I}-\mathrm{I_1I_2O})\,8\Delta &= abc\,(a+b) \end{aligned}\right\} \qquad (35)$$

$$\left.\begin{aligned} (2\,\mathrm{I_2I_3I}-\mathrm{I_2I_3H})\,8\Delta &= a(b+c)\,(b^2+c^2-a^2) \\ (2\,\mathrm{I_3I_1I}-\mathrm{I_3I_1H})\,8\Delta &= b(c+a)\,(c^2+a^2-b^2) \\ (2\,\mathrm{I_1I_2I}-\mathrm{I_1I_2H})\,8\Delta &= c(a+b)\,(a^2+b^2-c^2) \end{aligned}\right\} \qquad (36)$$

$$\left.\begin{aligned} \mathrm{I\,H\,O}\cdot 8\Delta &= s\,(c-b)\,(c-a)\,(a-b) \\ \mathrm{I_1H\,O}\cdot 8\Delta &= s_1(c-b)\,(c+a)\,(a+b) \\ \mathrm{I_2H\,O}\cdot 8\Delta &= s_2(b+c)\,(c-a)\,(a+b) \\ \mathrm{I_3H\,O}\cdot 8\Delta &= s_3(b+c)\,(c+a)\,(a-b) \end{aligned}\right\} \qquad (37)$$

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$$\left.\begin{aligned} &\mathrm{I\,I_1O}\cdot 8\triangle = (r_1-r)\,(r_3-r_2)\,(r_2r_3+rr_1)\\ &\text{and so on.}\end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} &\mathrm{I\,I_1H}\cdot 4\triangle = (r_1-r)\,(r_3-r_2)\,(r_2r_3-rr_1)\\ &\text{and so on.}\end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} &(2\,\mathrm{I_2I_3I}-\mathrm{I_2I_3O})\,8\triangle = (r+r_1)\,(r_2+r_3)\,(r_2r_3+rr_1)\\ &\text{and so on.}\end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} &(2\,\mathrm{I_2I_3I}-\mathrm{I_2I_3H})\,4\triangle = (r+r_1)\,(r_2+r_3)\,(r_2r_3-rr_1)\\ &\text{and so on.}\end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} &\mathrm{I\,H\,O}\cdot 32\triangle = \frac{1}{\mathrm{R}r}(r_1-r_2)(r_1-r_3)(r_2-r_3)(r_1-r)(r_2-r)(r_3-r)\\ &\mathrm{I_1H\,O}\cdot 32\triangle = \frac{1}{\mathrm{R}r_1}(r_1+r_2)(r_1+r_3)(r_3-r_2)(r_1-r)(r_2+r)(r_3+r)\\ &\text{and so on.}\end{aligned}\right\} \tag{42}$$

It is probable that some of the preceding 42 formulae may belong to earlier dates, and to other authors than those indicated below. I shall be glad if any reader, who knows of earlier sources than those I have recorded, will take the trouble to inform me.

(2), (3) Mr E. Hain in *Nouvelle Correspondance Mathématique*, I. 75 (1874-5).

(5), (6) Feuerbach, *Eigenschaften...des...Dreiecks*, §§ 8, 9 (1822).

(7), (8) Mr B. Möllmann in Grunert's *Archiv*, XVII., 396 (1851).

(9), (10) Mr Combier gives the first of each of these expressions in the *Journal de Mathématiques Élémentaires*, III., 351 (1879).

(11), (12), (14) Mr B. Möllmann in Grunert's *Archiv*, XVII., 393-6 (1851).

(13) C. J. Matthes gives the first expression in *Commentatio de Proprietatibus Quinque Circulorum*, p. 55 (1831); T. S. Davies gives the others in the *Lady's and Gentleman's Diary* for 1842, p. 87.

(15) The first expression is given by Mr A. R. Clarke in 1847 in the *Mathematician*, III., 45 (1856); all are given by Mr C. Hellwig in Grunert's *Archiv*, XIX., 43 (1852).

(17) These proportions are implied in Feuerbach, *Eigenschaften...des...Dreiecks*, § 61 (1822). The first of them is given by C. J. Matthes in his *Commentatio*, p. 55, and also the first of (18).

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(19) The first expression is given by C. Adams in his *Eigenschaften...des... Dreiecks*, p. 61 (1846): the corresponding expression for ABC is given on p. 62.

(20) The last values of each of these sets are given by Thomas Weddle in the *Lady's and Gentleman's Diary* for 1845, p. 75.

(21) T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 88.

(22)-(25) Mr Georges Dostor in the *Journal de Mathématiques Élémentaires et Spéciales*, IV. 21-23 (1880). The first expression for LMN, however, is given by Grunert in his article "Dreieck," quoted on p. 25. C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 15 (1825), gives the expression

$$\frac{2\Delta}{(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})-1}.$$

(26)-(28) Mr C. Hellwig in Grunert's *Archiv*, XIX., 25-26 (1852).

(30) Feuerbach, *Eigenschaften...des...Dreiecks*, § 23 (1822).

(31)-(32) C. Adams, *Eigenschaften...des...Dreiecks*, pp. 54, 53 (1846).

(33)-(42) Mr C. Hellwig in Grunert's *Archiv*, XIX., 43-50 (1852).

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Third Meeting, May 11th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., President, in the Chair.

The Nine-point Circle.
By DAVID MUNN, F.R.S.E.

An Account of Newton's "Opticks."
By CARGILL G. KNOTT, D.Sc., F.R.S.E.

Fourth Meeting, June 8th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., President, in the Chair.

New Proof of Professor Tait's Problem of Arrangement. Some short notes of interest to Teachers.
By THOMAS MUIR, M.A.

The Fundamental Notions of the Differential Calculus.
By A. Y. FRASER, M.A.

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Plücker's first equation connecting the singularities of Curves.

By Cargill G. Knott, D.Sc.

[The following notes are exactly as I left them in the hands of the Committee of the Society eleven years ago. They are printed now in the hope that, chiefly because of their brevity, they may be found useful to members, who may not have leisure or opportunity to read up the subject in the recognised text-books. —Jan., 1894. C. G. K.]

Let $U = f(x, y, z) = 0$ be the equation in trilinear co-ordinates of a curve of the n^{th} degree.

The number of terms is obviously

$$1 + 2 + 3 + \ldots\ldots + (n+1) = \tfrac{1}{2}(n+1)(n+2).$$

Hence $\frac{1}{2}(n+1)(n+2) - 1 = \frac{1}{2}n(n+3)$ points determine a general curve of the n^{th} degree.

According to Plücker, a curve may be considered as kinematically described by a point which moves along a line which continually rotates about that point.

This conception gives very simple notions as to the nature of singularities—double points, cusps, points of inflexion, double tangents, and so on.

Of these, only the cusp and point of inflexion are true singularities, the former being produced by a stationary point, the latter by a stationary line.

Still, a singular point may be more generally defined as a point which has descriptive properties not, in general, possessed by other points.

For example, a double point has two tangents, a triple point three, etc., a double tangent touches the curve in two distinct points, or, more strictly, meets the curve in two pairs of coincident points.

In the equation $U = f(xyz) = 0$

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substitute for x the expression $\lambda x + \mu x_1$, for y, $\lambda y + \mu y_1$, and for z, $\lambda z + \mu z_1$. Then by Taylor's theorem U becomes

$$\lambda^n U + \lambda^{n-1}\mu \nabla U + \lambda^{n-2}\mu^2 \frac{1}{\underline{|2}} \nabla^2 U + \ldots + \frac{\mu^n}{\underline{|n}} \nabla^n U$$

where $$\nabla = x_1 \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + z_1 \frac{\partial}{\partial z}.$$

This gives n values of λ/μ if equated to zero. That is, any arbitrary line cuts the curve in n points real or imaginary.

If the point xyz is on the curve as well as the point $(\lambda x + \mu x'$, etc.) then $U = 0$, and there are $(n-1)$ other points of intersection.

If, further, $\nabla U = 0$, then there are two coincident points $(\lambda/\mu = 0)$, and

$$\nabla U = x_1 \frac{\partial U}{\partial x} + y_1 \frac{\partial U}{\partial y} + z_1 \frac{\partial U}{\partial z} = 0$$

is the equation of the tangent at the point (xyz), $x_1\ y_1\ z_1$ being any point on the tangent.

If the point $x_1 y_1 z_1$ is fixed, the equation $\nabla U = 0$ of degree $(n-1)$ in xyz represents the First Polar, a curve of the $(n-1)^{th}$ degree cutting the curve

$$U = 0$$

in $n(n-1)$ points, which include all the points to which tangents can be drawn from the points $x_1 y_1 z_1$.

The greatest number of tangents which can be drawn from any point to the curve, is called the *class* of the curve.

In the general curve of the n^{th} degree the class is $n(n-1)$; but if singularities exist the class is not so great.

But the equation $\nabla U = 0$ may be true whatever $x_1 y_1 z_1$ may be *i.e.*, if

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial z} = 0 \text{ at } x, y, z.$$

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That is, any line through xyz cuts the curve in two coincident points, or in a double point.

Hence $\nabla U = 0$ must pass through all the double points.

Or the First Polar passes through the double points of a curve.

Hence, if there are δ double points, since each double point counts for two, the other intersections cannot be greater than

$$n(n-1) - 2\delta,$$

and this is of course a superior limit to the number of tangents than can be drawn from a given point to the curve.

Consider more closely the conditions

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial z} = 0.$$

Here there is, generally speaking, if $\nabla^2 U$ does not identically vanish, a node. The particular character of the node is got by considering the properties of $\nabla^2 U$.

If $\nabla^2 U = 0$ as well as $U = 0$, $\nabla U = 0$, there are three coincident points.

For a double point the equation $\nabla^2 U = 0$ gives two lines—two values of $x_1 y_1 z_1$.

In certain cases, however, these tangent lines coincide, namely, when

$\nabla^2 U$ is a complete square,

i.e., when

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = Z = 0$$

$$\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial^2 u}{\partial y \partial z}\right)^2 = X = 0$$

$$\frac{\partial^2 u}{\partial z^2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial^2 u}{\partial z \partial x}\right)^2 = Y = 0.$$

But in this case the tangent to any First Polar

$$\nabla U = 0$$

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is given by the condition

$$\left(x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right) \nabla U = 0$$

independently of the values $x_1 y_1 z_1$ in ∇U

or

$$\begin{vmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial x \partial z} \\ \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial y^2} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = 0 = H$$

must be satisfied. But evidently $X = 0$, $Y = 0$, $Z = 0$ satisfy this.

Hence at a cusp every first polar passes through the curve and has the same tangent with it—that is, meets the curve in three coincident points.

Hence if there are κ cusps, each cusp counts for three points, and the other points of intersection, or, what is the same thing, the *class* of the curve, is

$$n(n-1) - 2\delta - 3\kappa.$$

The equation $H = 0$ is called the Hessian. Its order is $3(n-2)$. It intersects the curve in $3n(n-2)$ points; and if there are no nodes or cusps these are points of inflexion of the original curve and the Hessian meets the curve in three coincident points.

Hence, generally, if ι = number of inflexions

$$\iota = 3n(n-2).$$

But if there are nodes and cusps, this number ι is reduced. The analytical condition for the Hessian is

$$U = 0, \quad \nabla U = 0, \quad \nabla^2 U = 0.$$

But for a double point U and ∇U vanish independently of $x_1 y_1 z_1$; and the polar conic $\nabla^2 U = 0$ reduces to the two tangents at

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the double point; hence with these conditions $x_1y_1z_1$ may be a point anywhere on either of the tangents. ∇U vanishes because

$$\frac{\partial U}{\partial x}=0 \quad \frac{\partial U}{\partial y}=0 \quad \frac{\partial U}{\partial z}=0$$

or

$$\left.\begin{aligned} x\frac{\partial^2 U}{\partial x^2}+y\frac{\partial^2 U}{\partial x\partial y}+z\frac{\partial^2 U}{\partial x\partial z}&=(n-1)\frac{\partial U}{\partial x}=0 \\ \ldots \text{ etc. } \ldots \quad \ldots \quad \ldots \quad &=0 \\ \ldots \text{ etc. } \ldots \quad \ldots \quad \ldots \quad &=0 \end{aligned}\right\}$$

Whence eliminating xyz

$$H=0, \text{ the Hessian.}$$

Or the Hessian also passes through all the double points.

But, the general condition of there being three coincident points in which H intersects U obviously requires that H must touch U there, must have the two tangents also two tangents, must itself have a double point there.

Hence a double point counts for 6 intersections between H and U.

For rigid proof, take the node as origin. Hence U contains no term in x and y lower than the second degree. Consider the lowest dimensions of $H=0$ in x and y, thus:

$$\begin{vmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x\partial y} & \frac{\partial^2 U}{\partial x\partial z} \\ \frac{\partial^2 U}{\partial x\partial y} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial y\partial z} \\ \frac{\partial^2 U}{\partial x\partial z} & \frac{\partial^2 U}{\partial y\partial z} & \frac{\partial^2 U}{\partial z^2} \end{vmatrix} \text{ is } \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Hence the order of the lowest terms in x and y is 2. Therefore $H=0$ has also a double point at the origin.

Again, let the tangent at the origin be the line x. Then x will be a factor in the equation $U=0$ and must be present in the expressions $\frac{\partial^2 U}{\partial y^2}, \frac{\partial^2 U}{\partial z^2}, \frac{\partial^2 U}{\partial y\partial z}$, one of which is present in every term of $H=0$. Hence x is also a tangent to H.

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Hence if there are δ nodes

$$\iota = 3n(n-2) - 6\delta.$$

Take now the case of the cusp. Take it as origin and let $x=0$ be the tangent. Then x^2 is a factor in $U=0$. Hence the lowest dimensions in x and y of H are

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix}$$

Hence 3 is the order of the lowest term in $H=0$, and each term contains x^2 as a factor. Hence this point is a triple point on H formed by a simple branch passing through a cusp; and the coincident tangents coincide with the cuspidal tangent. But a triple point and a double point meet in 6 coincident points, and the common pair of tangents means other two points. Hence H cuts U at a cusp in 8 points.

Hence if there are δ nodes and κ cusps

$$\iota = 3n(n-2) - 6\delta - 8\kappa.$$

The possible double points to the curve are limited in number; they cannot exceed $\frac{1}{2}(n-1)(n-2)$. For, if possible, let there be $\frac{1}{2}(n-1)(n-2)+1$.

Then, these points together with $(n-3)$ other points will determine a curve of degree $(n-2)$ because these together give

$$\tfrac{1}{2}(n-2)(n+1) = \tfrac{1}{2}\{n-2\}\{n-2+3\}.$$

Hence this curve of degree $(n-2)$ intersects the curve of degree n in

$$(n-1)(n-2)+2+n-3 \text{ points}$$
$$= (n-1)^2 = n(n-2)+1$$

which is impossible.

The number

$$D = \tfrac{1}{2}(n-1)(n-2) - \delta - \kappa$$

is called the "deficiency" of the curve. It is of great importance in the theory of transformations.

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The curve for which $D = 0$ is unicursal.

The curve for which $D = 1$ is bicursal, and so on.

Fifth Meeting, July 13th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., President, in the Chair.

Some notes on Quaternions.

By CARGILL G. KNOTT, D.Sc., F.R.S.E.

Some theorems on Radical Axes.

By DAVID MUNN, F.R.S.E.

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Edinburgh Mathematical Society.

LIST OF MEMBERS.

FIRST SESSION, 1883.

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C. W. C. Barlow, M.A., St Peter's College, Cambridge.

James Blyth, M.A., F.R.S.E., Professor of Mathematics and Natural Philosophy, Andersonian College, Glasgow.

5 James Bolam, Navigation School, Leith.

H. H. Browning, M.A., 14 Newton Place, Charing Cross, Glasgow.

J. R. Burgess, M.A., Mathematical Master, Merchiston Castle School, Edinburgh.

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PETER CAMPBELL, Mathematical Master, George Watson's College, Edinburgh.

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J. MACDONALD, M.A., 11 South Charlotte Street, Edinburgh.

W. J. MACDONALD, M.A., Mathematical Master, Daniel Stewart's College, Edinburgh.

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CHARLES M. MIDDLETON, M.A., Mathematical Master, George Watson's College, Edinburgh.

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R. T. OMOND, The Observatory, Ben Nevis.

WILLIAM PEDDIE, Assistant to the Professor of Natural Philosophy, University, Edinburgh.

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THOMAS T. RANKIN, C.E., Rector, Gartsherrie Science School, Coatbridge.

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GEORGE THOM, M.A., Rector, Dollar Institution.

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Sir WILLIAM THOMSON, D.C.L., F.R.S., &c., Professor of Natural Philosophy, University, Glasgow, (*Hon. Member*).

WILLIAM THOMSON, M.A., Student in Divinity, Edinburgh.

DAVID TRAILL, M.A., B.Sc., Mathematical Master, Academy, Edinburgh.

JOHN E. VERNON, 11 Duncan Street, Edinburgh.

55 WILLIAM WELSH, M.A., Fellow and Tutor, Jesus College, Cambridge.

THOMAS WHITE, 7 Claremont Place.

LAURENCE H. WIGHT, M.A., Lauder Road, Edinburgh.

58 PHILIP WOOD, M.A., Head Master, Grammar School, Darlington.

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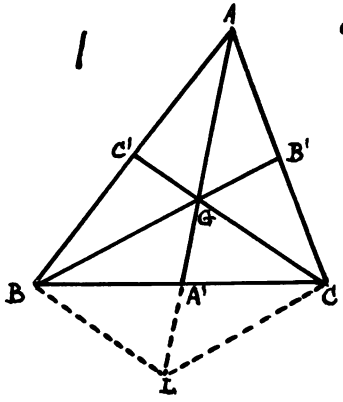

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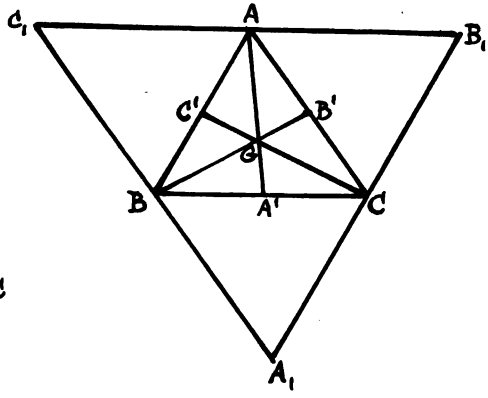

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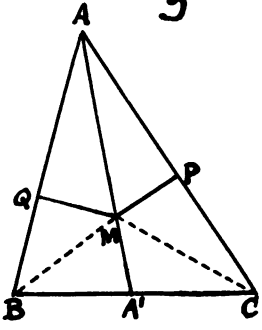

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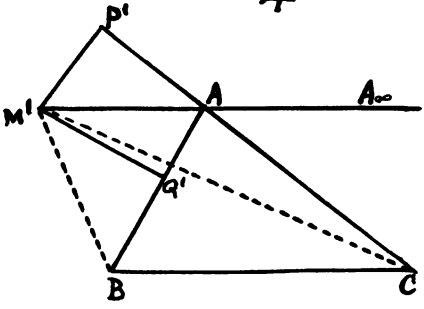

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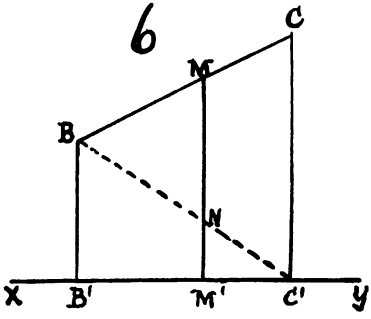

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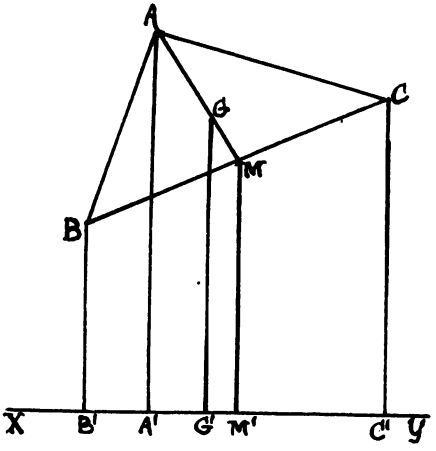

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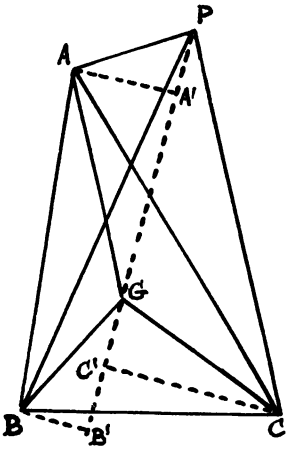

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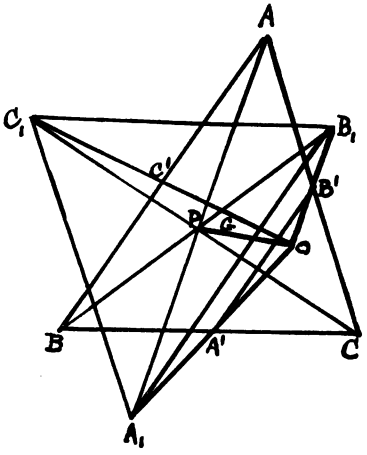

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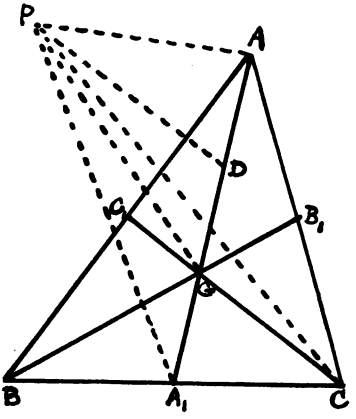

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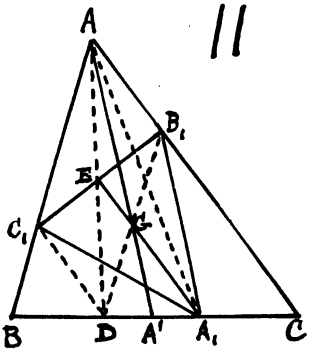

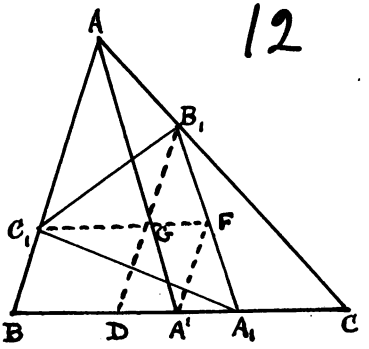

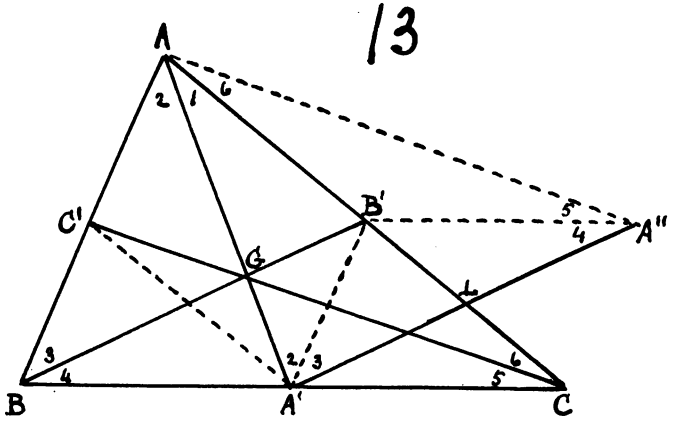

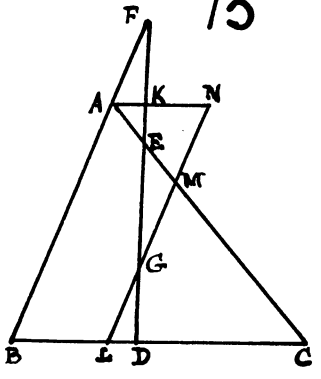

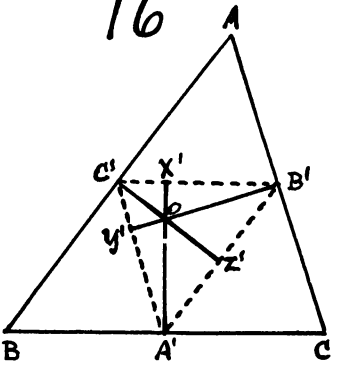

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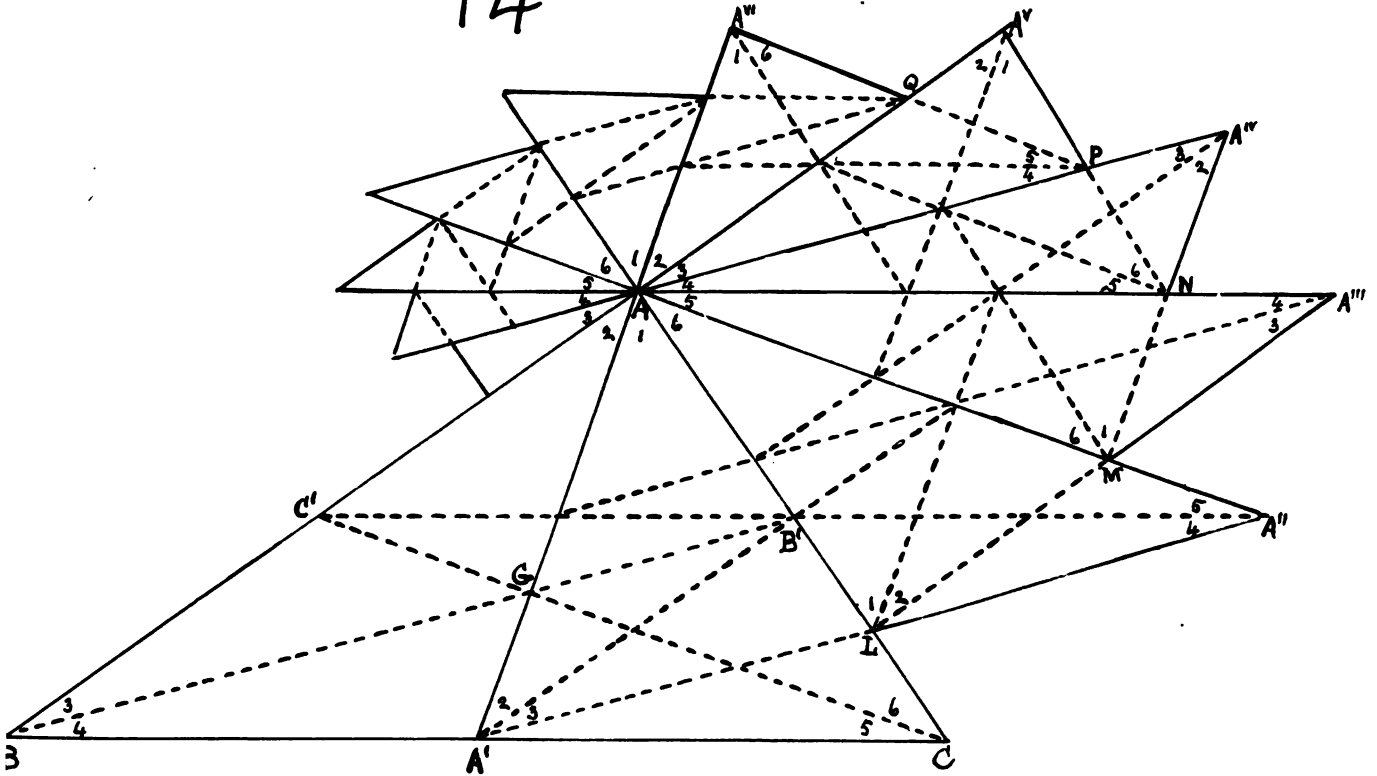

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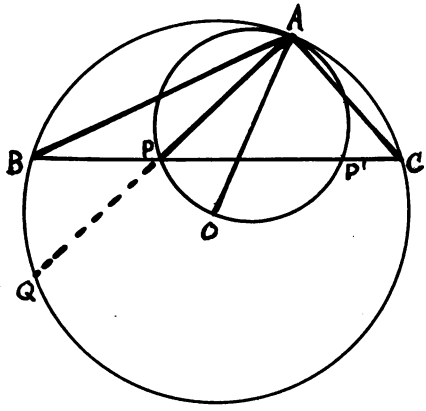

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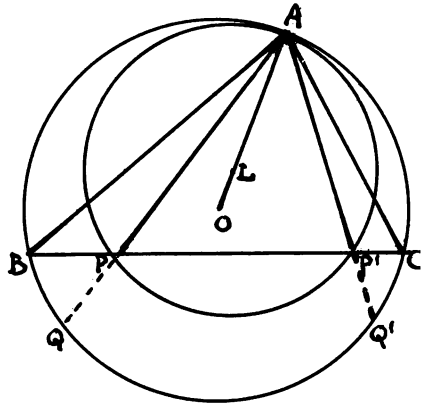

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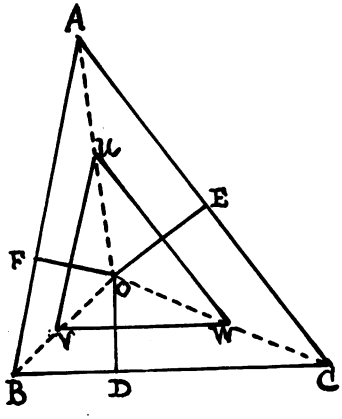

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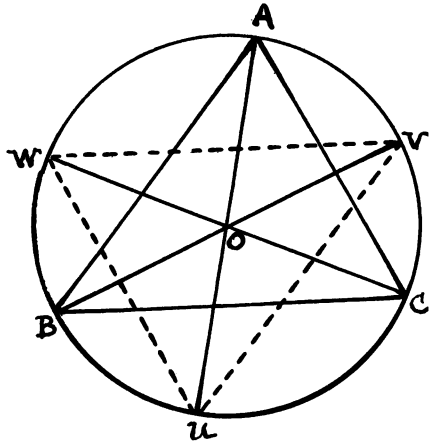

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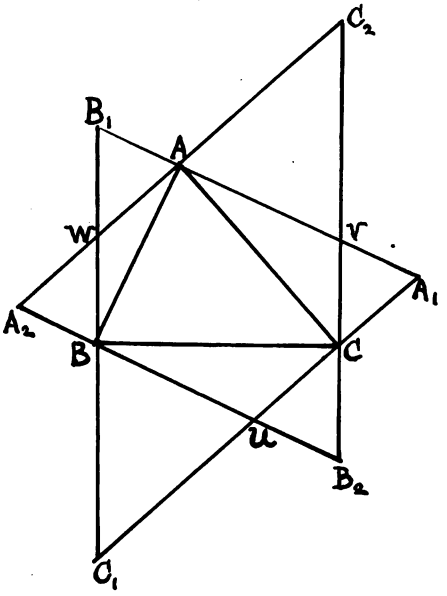

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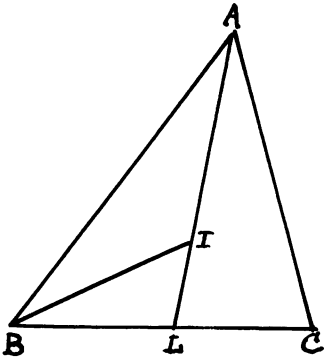

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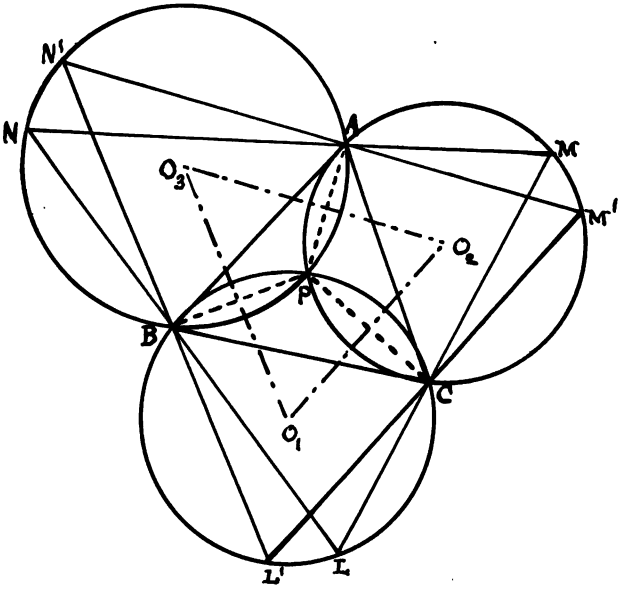

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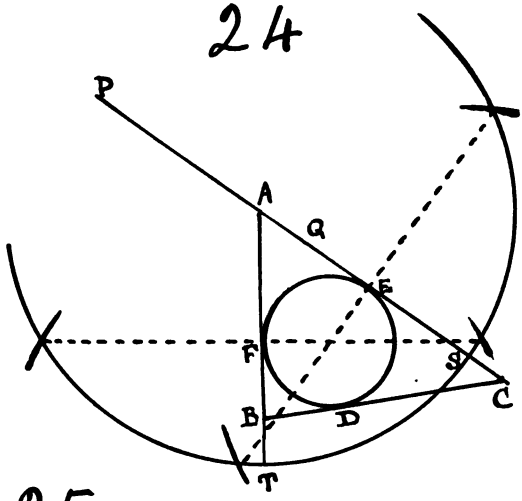

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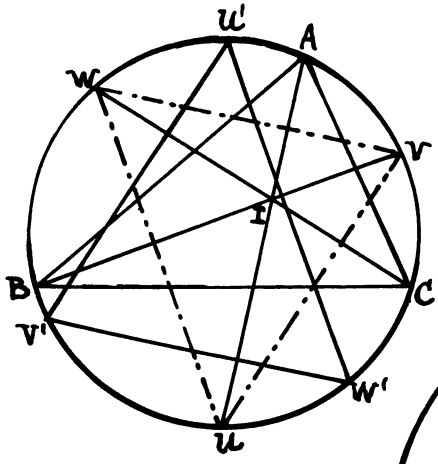

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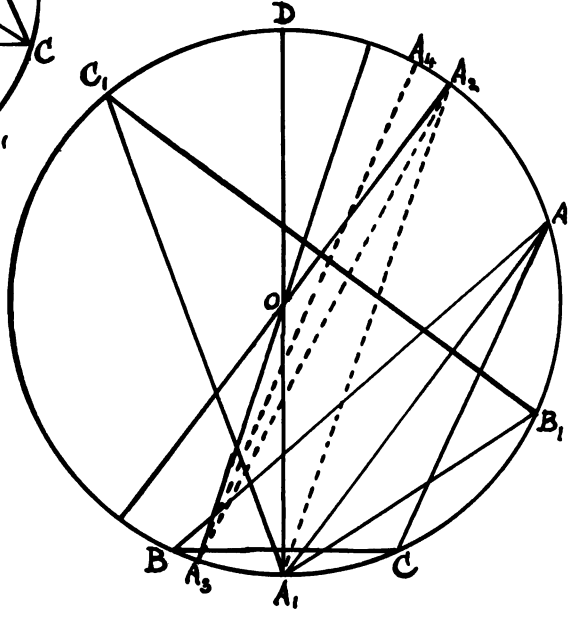

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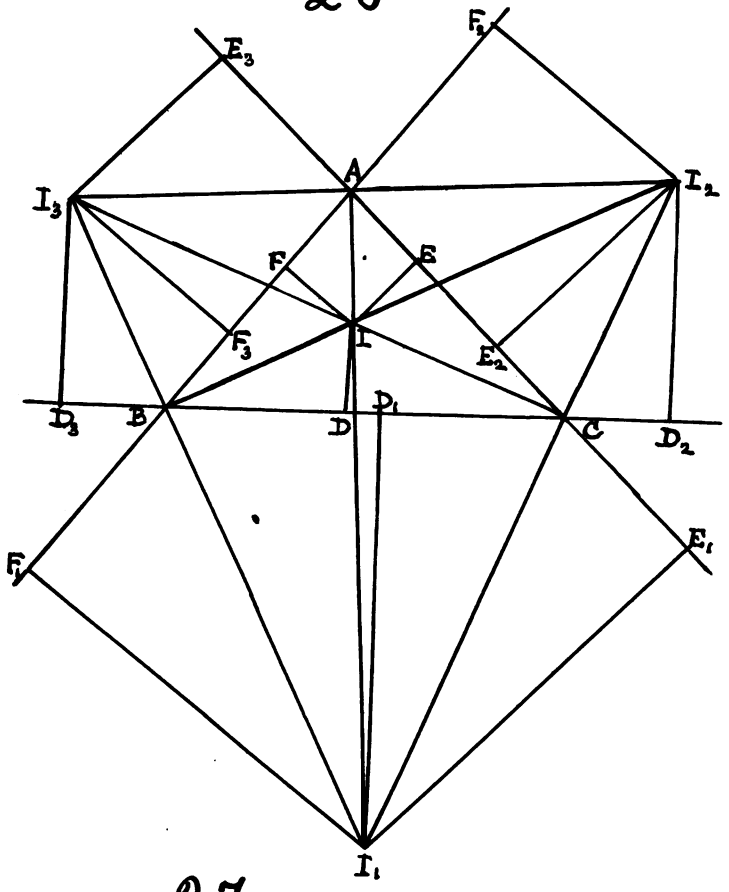

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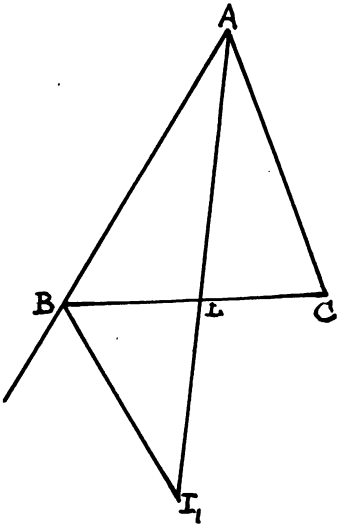

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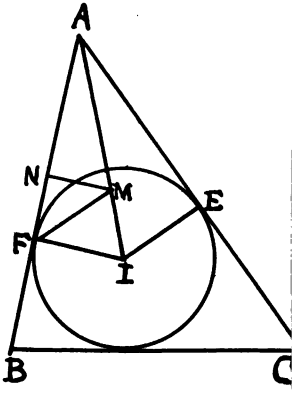

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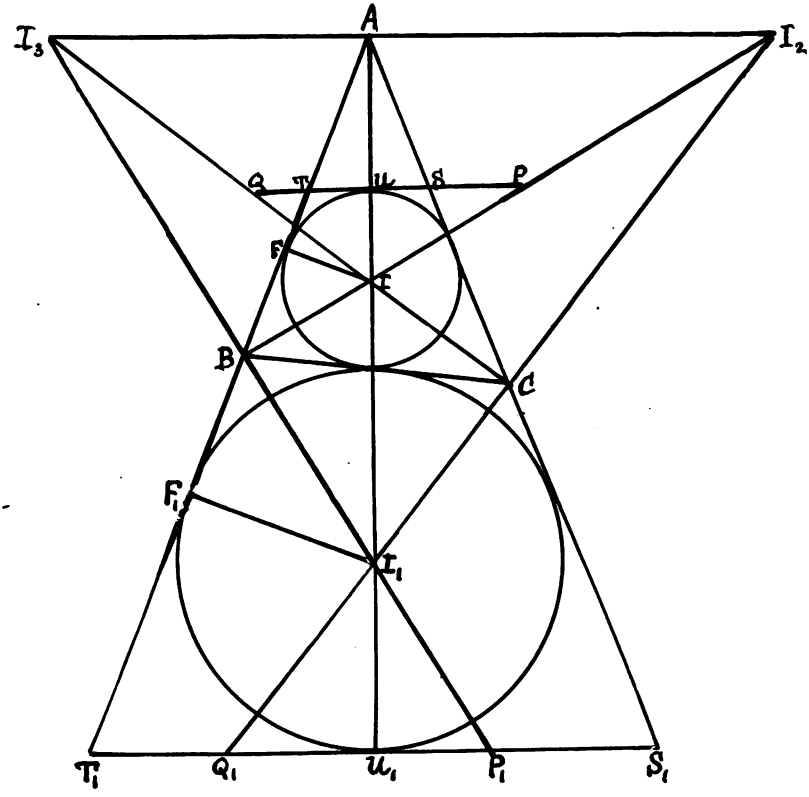

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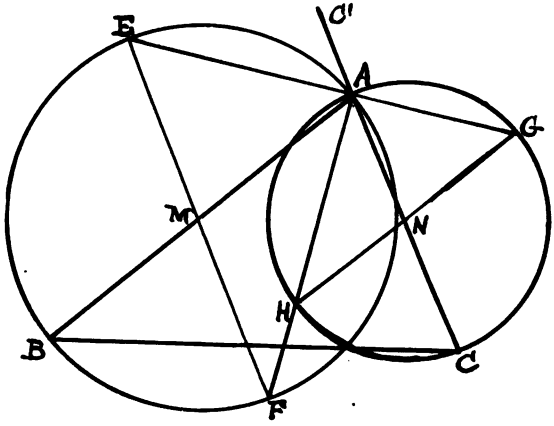

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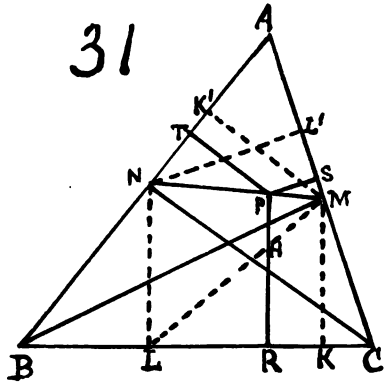

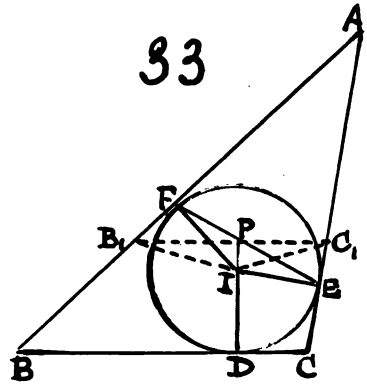

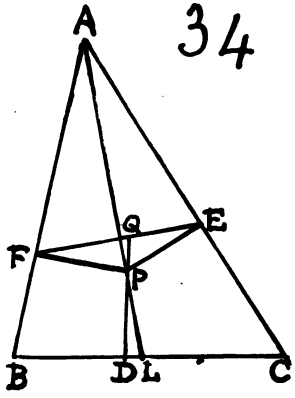

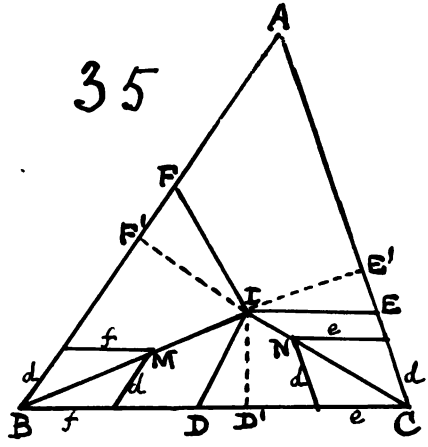

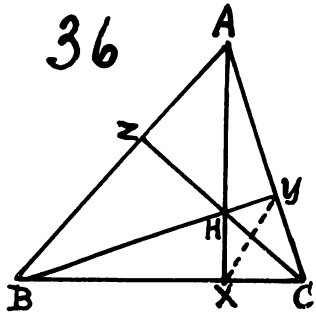

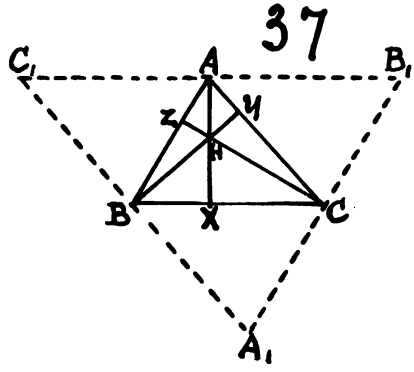

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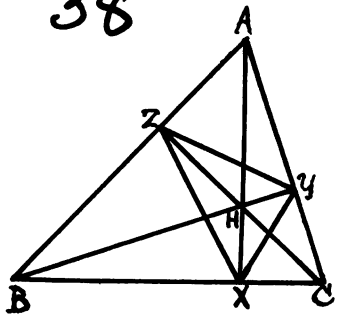

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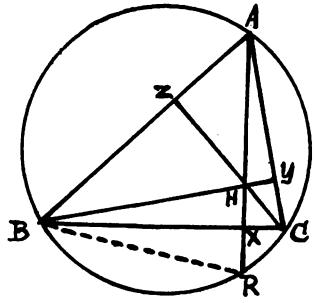

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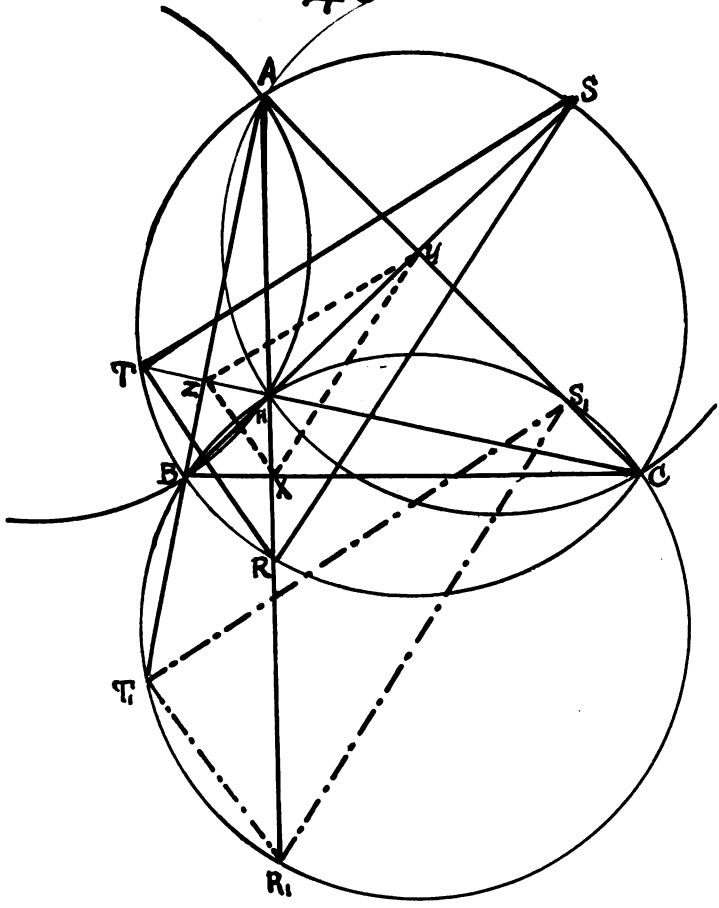

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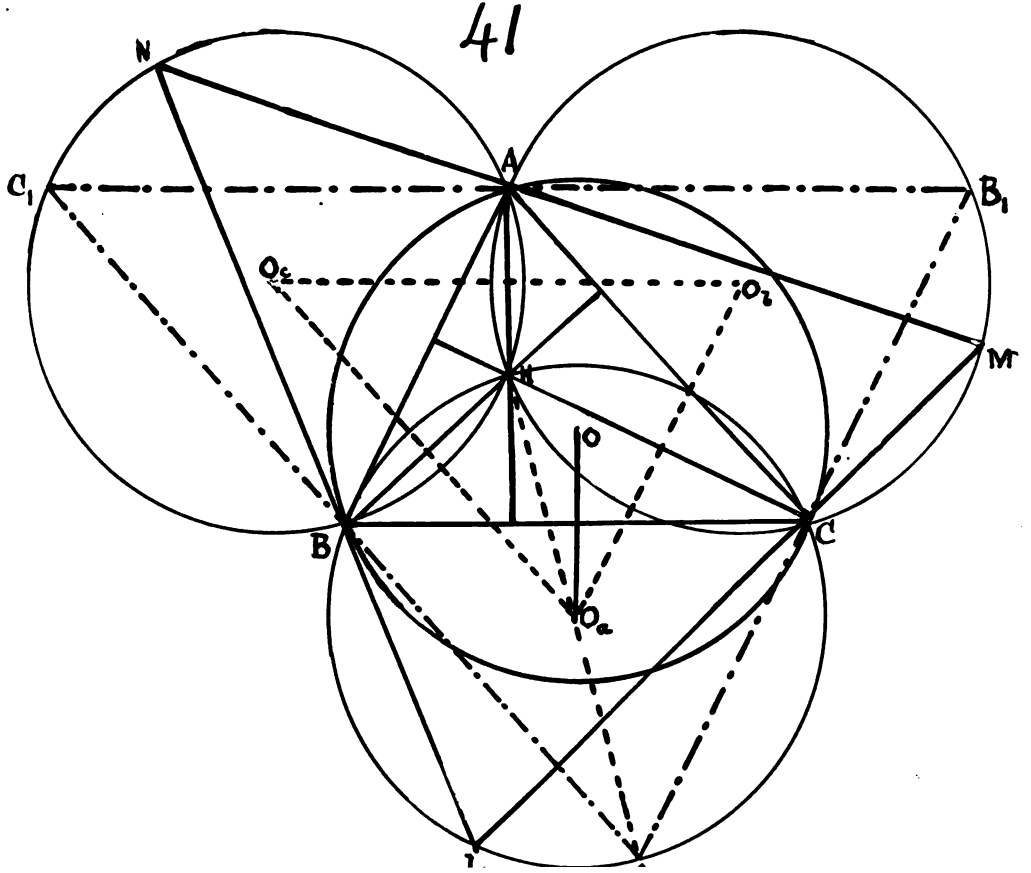

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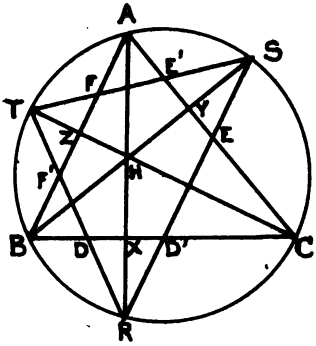

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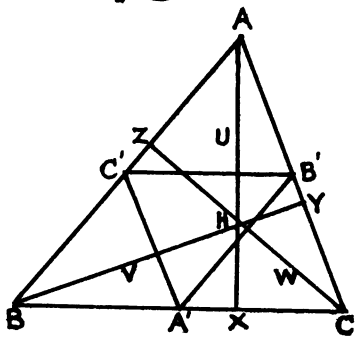

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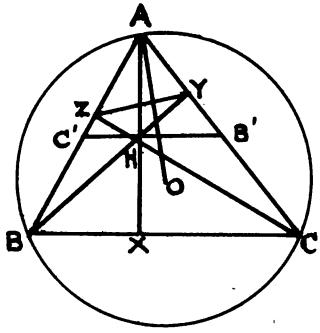

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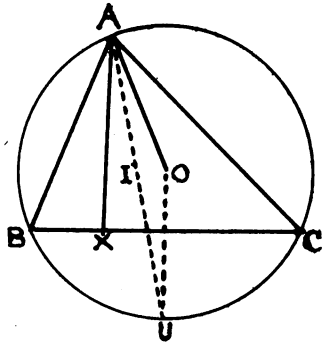

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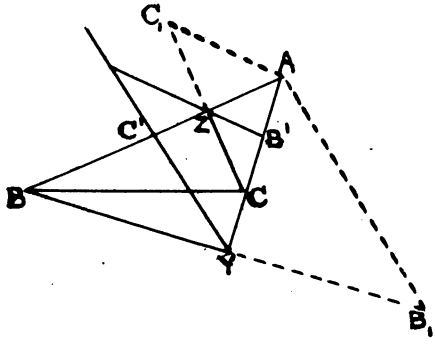

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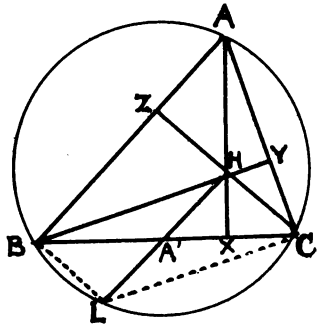

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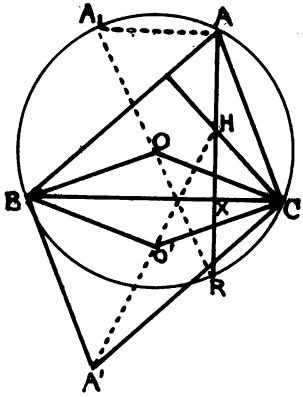

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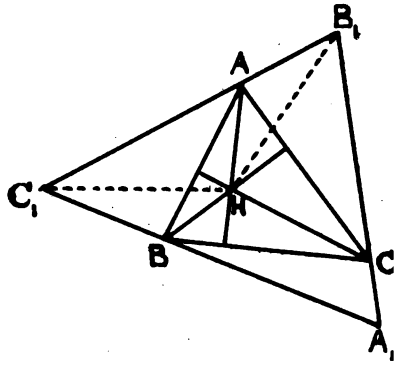

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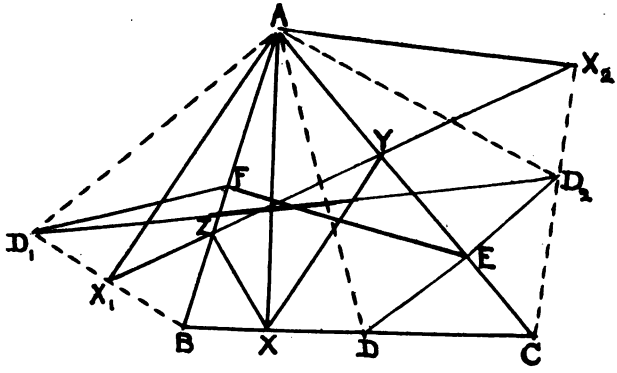

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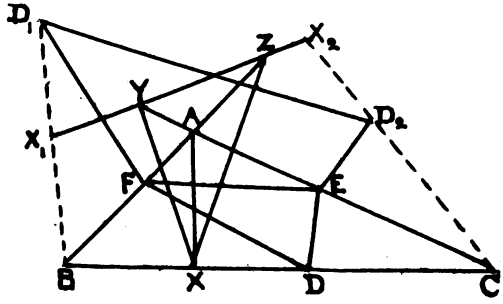

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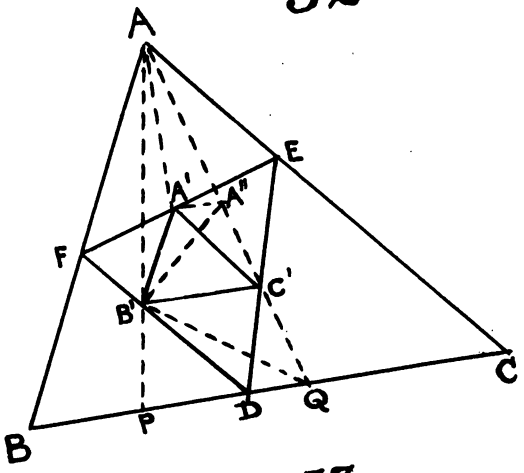

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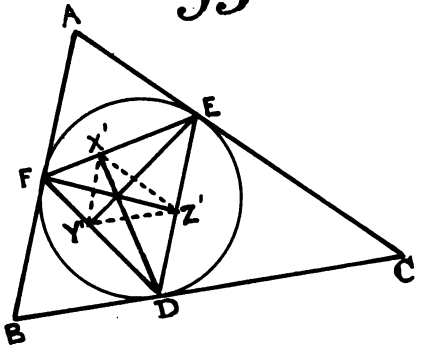

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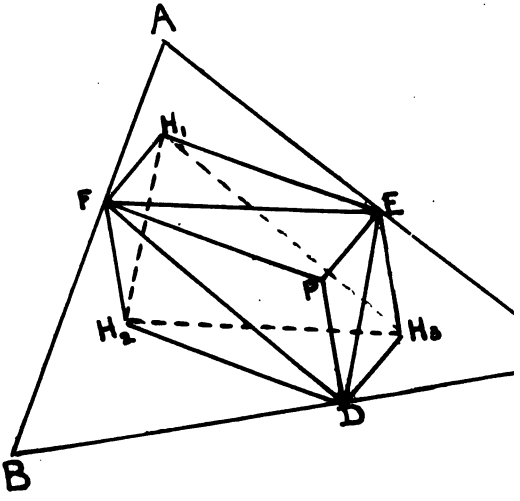

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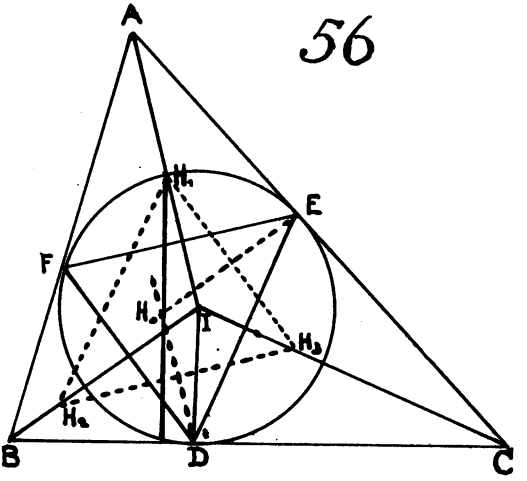

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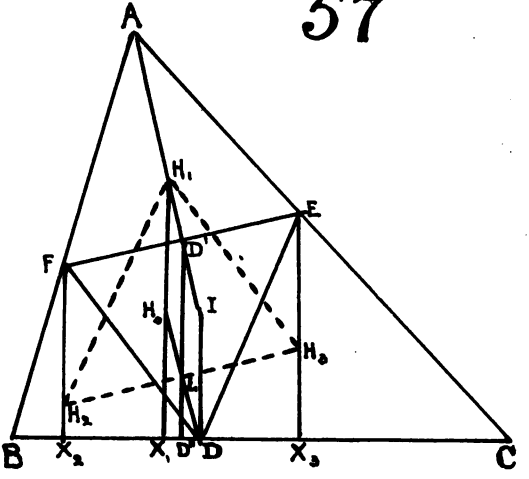

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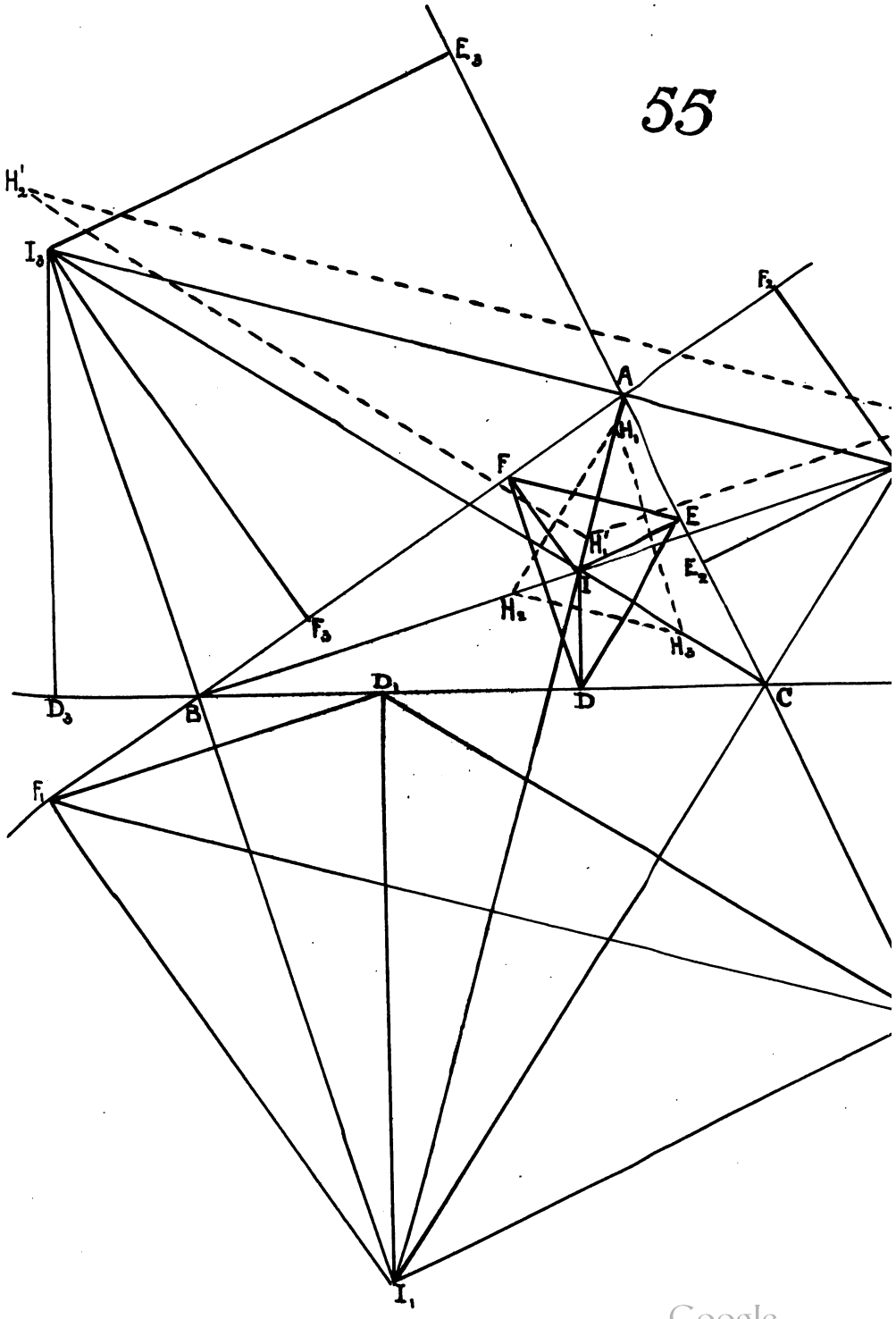

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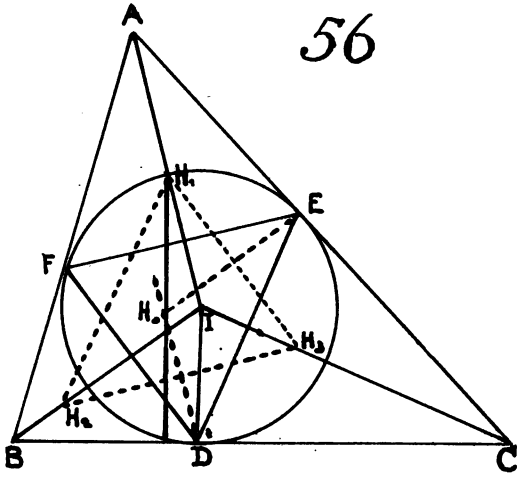

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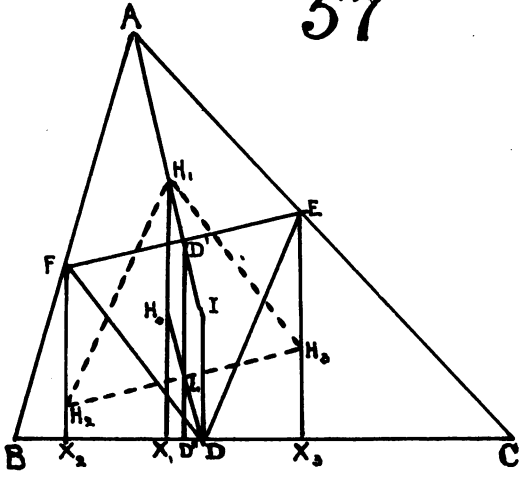

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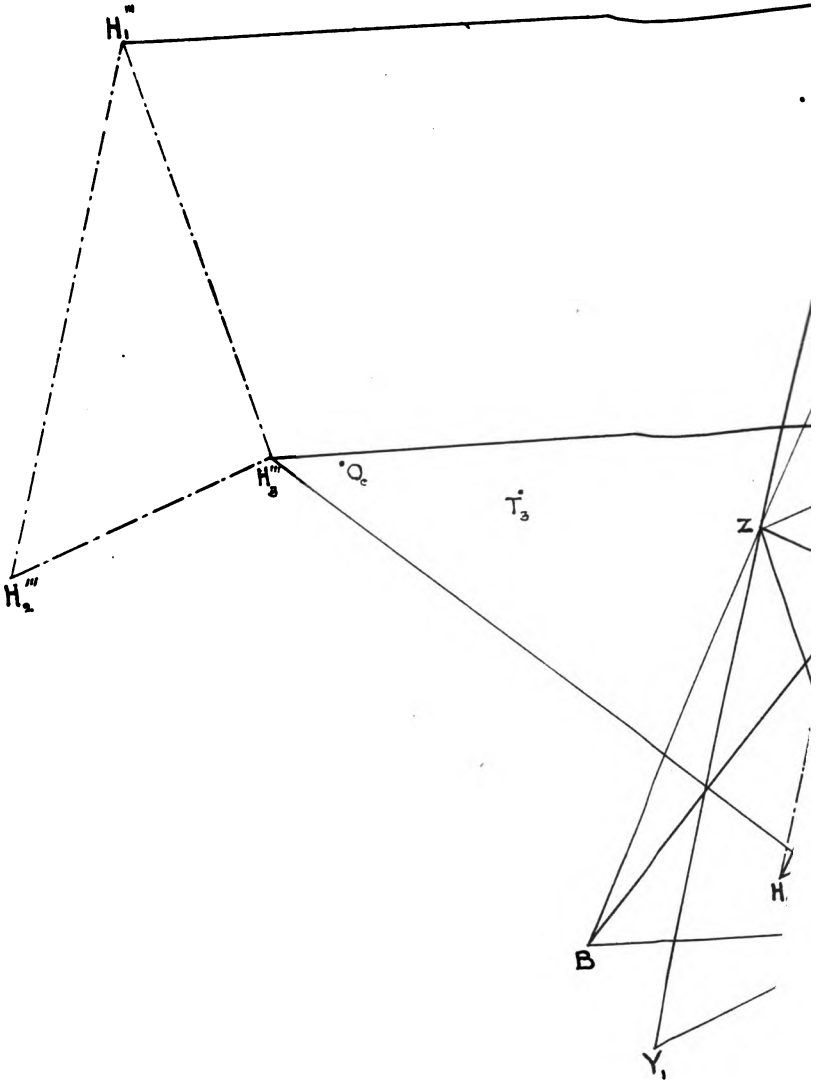

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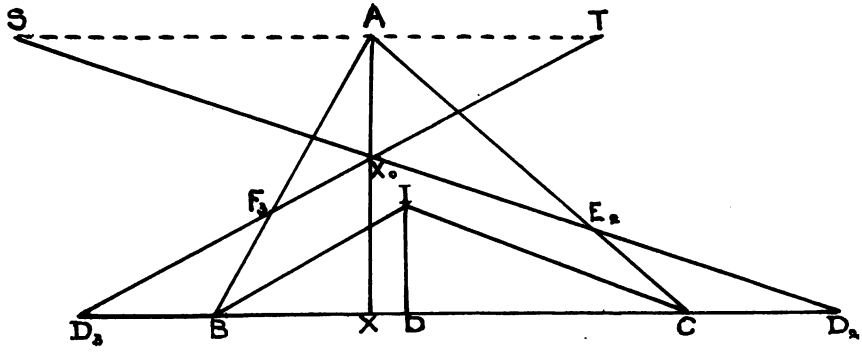

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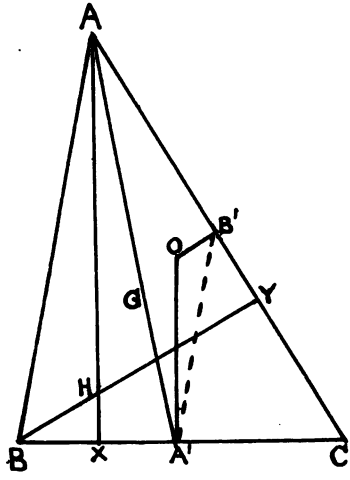

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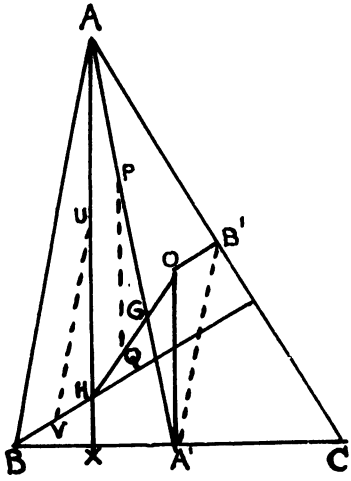

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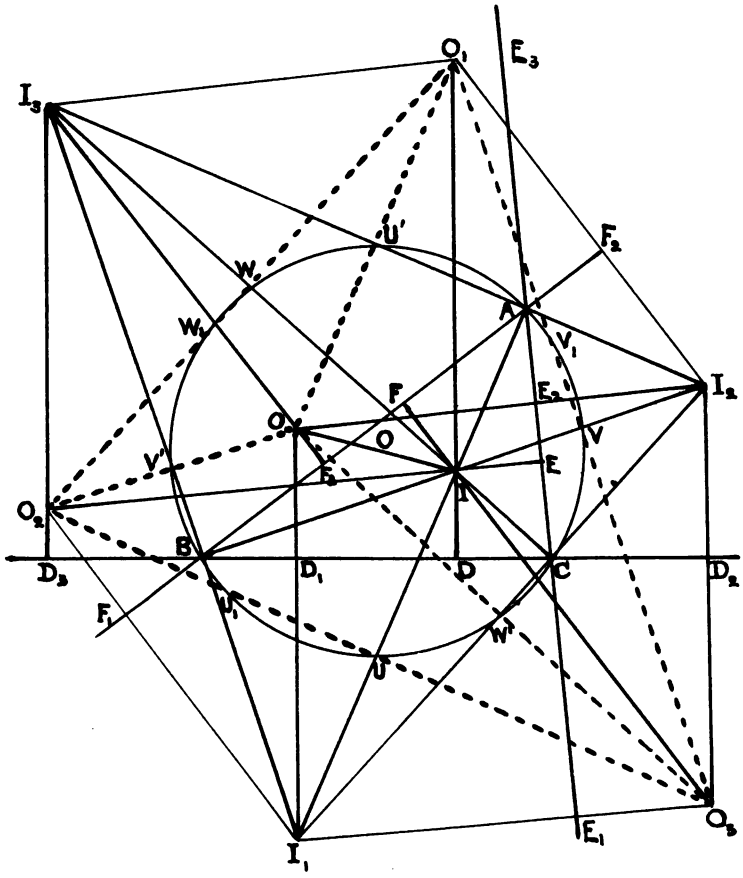

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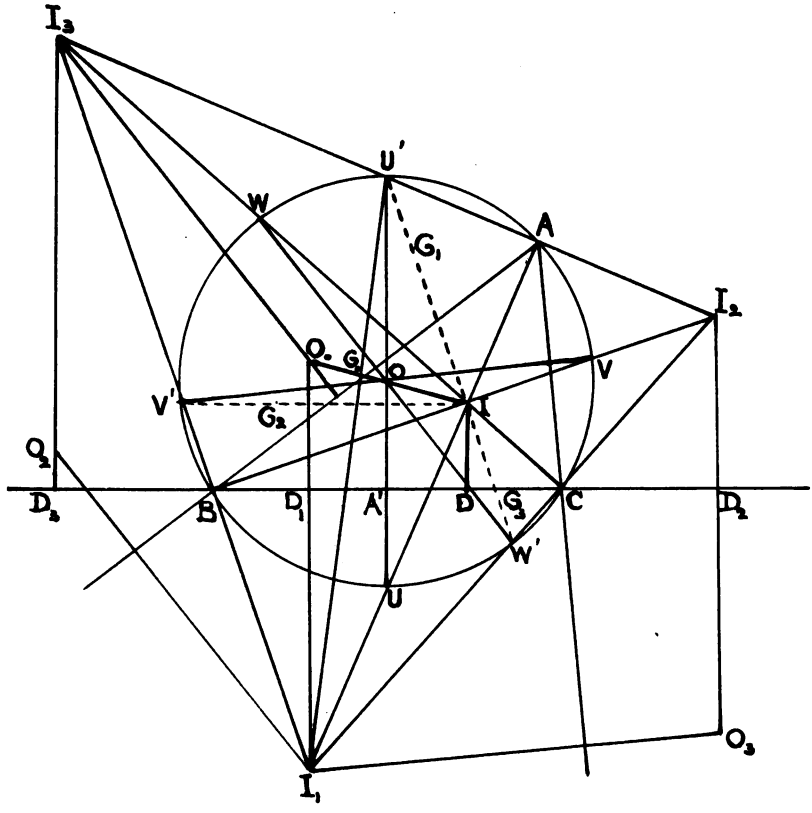

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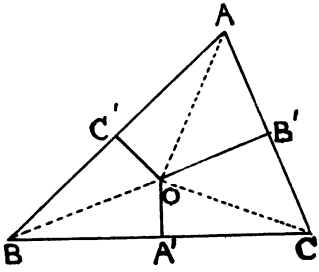

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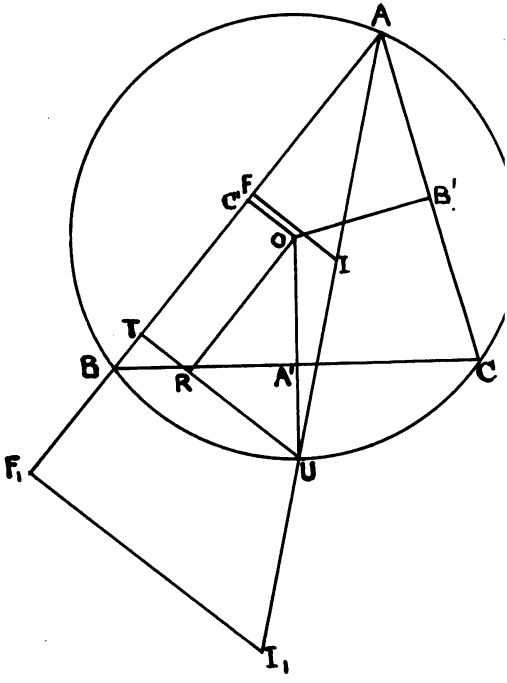

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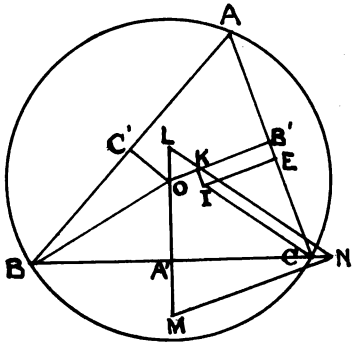

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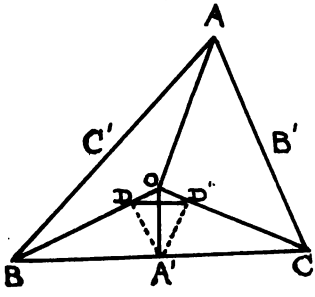

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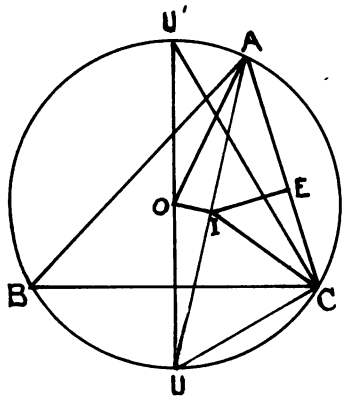

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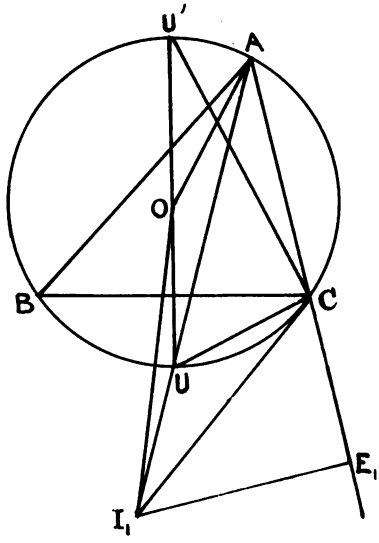

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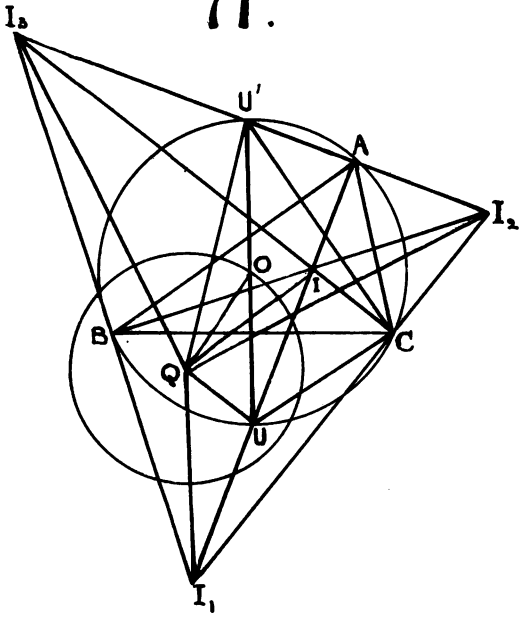

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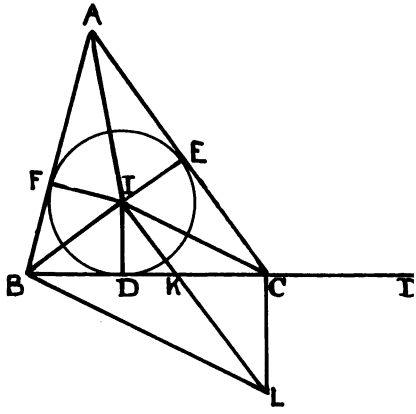

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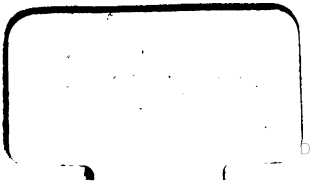

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